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Optimal network implementable controllers for networked systems

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The student author, whose presentation of the scholarship herein was approved by the program of study committee, is solely responsible for the content of this dissertation. The Graduate College will ensure this dissertation is globally accessible and will not permit alterations after a degree is conferred.

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DEDICATION

I would like to dedicate my dissertation to four precious people in my life. My beloved son Hamza who has put meaning into my life, my dearest husband who has supported me during my studies, and finally to my loving mother and father who helped me get to where I am today with their unlimited support. I would like to thank all of my family for their prayers, emotional support and encouragements. I will always appreciate all they have done.

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ABSTRACT

In this thesis, we study the problem of network implementable controllers for network distributed systems. Network distributed control problem gains importance by the increase in networked system applications in many areas which require network distributed control and estimation. By network implementable controller, we mean controller can be implemented over the given network with the predefined/given delay and sparsity constraints.

We define all stabilizing controllers by re-interpreting plant and controller. We define a congruent stable plant of the original plant which is not necessarily stable, such that the controller of the congruent plant is linearly function of the original plant's controller. When we put structural constraints on all stabilizing controllers of the stable congruent plant, these controllers embody controllers of the main plant. Therefore, all stabilizing controllers of the original plant are defined as all stabilizing controllers of the congruent plant with structural constraints. In the view of this problem, we obtain all stabilizing controller parametrization of the original plant wherein equality constraints are introduced on the Youla parameter. Moreover, we define a necessary and sufficient problem to attain a controller in the form of norm minimization problem benefiting formulated all stabilizing controller parametrization and provide a solution method for it.

Moreover, we introduce a doubly-coprime factorization of blkdiag(I_{n_x}, K) which allows us to have a network implementable state-space realization of a structured controller, K, which inherits sparsity and delay constraints introduced by the given network in z-domain, of a network distributed system with order n_x. By network implementable state-space realization, we mean state-space realization can be expressed as a strictly causal interaction of some sub-systems over the given network. We call such structured controllers as network realizable controller, i.e. controllers whose network implementable state-space realization

can be obtained. Moreover, using the formulated controller problem, we provide a network realizable controller problem by introducing sparsity and delay constraints on the Youla parameter. Introduced network realizable controller problem is in the form of norm minimization problem with structural constraints introduced on Youla parameter. Afterwards, we obtain its equivalent unconstrained network realizable controller problem which allows us to attain a solution in infinite dimensional space benefiting existing solution methods of $\mathcal{H}_2$ problem.

Moreover, we define a model matching problem and present an optimal network realizable controller problem. The formulated optimal network realizable controller problem is a constrained problem. To obtain an unconstrained problem formulation, we define a relaxation by a Lagrange multiplier and benefit from the vectorization method introduced in the literature. Formulated unconstrained problem allows us to obtain a solution using existing solution methods wherein solution lies in infinite dimensional space. Once the optimal network realizable controller is obtained, we obtain a network implementable state-space realization of it using the method we have introduced.

Furthermore, we provide an alternative all stabilizing network realizable controller parametrization benefiting existing Youla parametrization which requires to have an initial controller. We show that when the given initial controller is network realizable, one can parametrize all stabilizing network realizable controllers with a network realizable Youla parameter. Moreover, we introduce network realizable controllers in the form of delayed controllers for strongly connected networked plants which allow us to parametrize all stabilizing network realizable controllers with the Youla parametrization aforementioned. We derive a model matching problem and define a necessary and sufficient optimal network realizable controller problem as a function of initial network realizable controller with sparsity and delay constraints introduced on Youla parameter. Moreover, we provide its equivalent unconstrained problem benefiting vectorization method wherein a solution in infinite dimensional space can be obtained benefiting existing solution methods.

CHAPTER 1. INTRODUCTION

With the increase in size of network systems, their topological constraints and communication limitations bring challenges in network control problems. The networked system can be very large, so that communicating with the whole network could be cumbersome, or communication restrictions may exist in the network. In the platooning of cars, network distributed heating systems, network distributed controllers has started to gain importance due to large network sizes.

In this thesis, we are interested in solving optimal network distributed controller problem with necessary and sufficient conditions in infinite dimensional space and further obtaining a network implementable state-space realization of it over the given network. There exist numerous studies on design of network distributed controllers which impose network constraints on the controller transfer functions. Some of these works can be given as network distributed controllers for spatially invariant systems [2], [6], [41], systems with triangular and band structures [42]-[30], symmetrically interconnected systems [14], dynamically coupled systems [10], poset causal systems [35], and in the case of plant and controller structures satisfying quadratic invariance property [23], [33].

Furthermore, optimal state-feedback controller for network distributed systems have been formulated in [24] by utilizing ADMM method. In the work of [34], decentralized controllers have been designed using LMI approach. In [22], network distributed controllers have been designed for systems interconnected over an arbitrary graph using distributed LMIs which achieve a $\mathcal{H}_\infty$ performance. In [25] and [13], network distributed controllers have been studied for heterogeneous and heterogeneous linear parameter varying systems, respectively. In [25], a network distributed controller problem for heterogeneous dynamically coupled systems has been provided, based on $\mathcal{L}_2$ gain performance using bilinear matrix inequalities. In [13], network distributed linear parameter varying (LPV) controllers for the control of heterogeneous LPV systems have been shown. Additionally, there exist network distributed controller design problems which utilize the LMIs by involving the state-space parameters of the plants and controllers [26, 36]. Networked distributed control for identical dynamically coupled systems has been analyzed in [26]. In the work of [9], a sequential convex problem has been formulated to find sparse $\mathcal{H}_2$ optimal state-feedback controllers. Moreover, for positive systems, a network distributed controller problem is designed in [32]. Moreover, a model based network distributed controller has been obtained starting with network realizable dynamic state feedback controller and dynamic observer in [29].

To address the limited amount of local information in network distributed systems, various network distributed design methods have been formulated to have a stabilizing network distributed controllers. [12, 11] have studied the model predictive control using gradients method and control Lyapunov functions. [31] solves the minimization problem using dynamic dual decomposition which allows networked systems to solve the problem in a network distributed fashion; however, it does not guarantee stability for network distributed network problems. Moreover, [7] develops a state-feedback controller for continuous systems based on a gradient method which solves the infinite horizon linear quadratic cost functional reformulated by a terminal cost term where the systems are assumed to be stabilizable by a diagonal state feedback controller. In [20], a distributed LMI problem has been formulated to have full order network distributed controllers in a distributed fashion.

In literature, optimal network implementable controllers have been studied with sufficiency conditions benefiting existing Youla parametrization [1]. Furthermore, [43] provides an optimal control problem which can be solved in a convex way without depending on well-known Youla parametrization. A potential drawback is that stabilization is posed as infinite dimensional constraints with not know a priory finite support solution. To the authors' knowledge, necessary and sufficient conditions for network implementable $\mathcal{H}_2$ optimal controller problem to have a solution in infinite-dimensional space has not been studied so far.

Existing all stabilizing controller parameterizations require to have either doubly coprime factorization of plants or to have an initial output feedback controller. To refrain from this prerequirements, in chapter 3, we obtain all stabilizing controller parameterization with constraints introducd on Youla parameter. All stabilizing controller problem is well studied problem in the literature. Internally stabilizing controller for linear systems have been parametrized by well-known Youla parameter [44] which has showed that closed loop response can be shaped by a stable parameter. Afterwards, a closed loop parametrization for discrete time systems have been shown in [18]. Later, various all stabilizing controller problems have been formulated benefiting affine Youla parameter to achieve various specifications. Achievable closed loop maps has been parametrized using stable factorizations in [39]. State-space formulas has been derived in [8] for all stabilizing controllers solving standard $\mathcal{H}_\infty$ and $\mathcal{H}_2$ problems. Moreover, all stabilizing controller parametrization has been provided using doubly-coprime factorization of plant in [5] wherein the closed loop optimization is also addressed. In [27], all stabilizing controllers has been parametrized as a function of an initial output feedback controller. Set of all $\mathcal{H}_\infty$ controllers explicitly parametrized in the state-space using solutions of linear matrix inequalities [16]. Moreover, [3] utilizes linear matrix inequalities to obtain output feedback controllers with $\mathcal{H}_\infty$ and $\mathcal{H}_2$ performances. Nonlinear state-feedback $\mathcal{H}_\infty$ controllers and internally stabilizing optimal controllers for nonlinear systems have been presented in [38] and [15], respectively. Furthermore, [43] provides an optimal control problem which can be solved in a convex way without depending on well-known Youla parametrization.

We formulate all stabilizing controller problem by interpreting plant and controller from a different perspective. We define a congruent plant of any stable/unstable plant, P, as $\bar{P}$ which is stable, such that it is defined from $[i^T; u^T]^T$ to $[o^T; y^T]^T$ and when input channel-i

connects with output-channel-o with a unity feedback, map from u to y is equivalent to P_{22}. Therefore, we define a controller of $\bar{P}$ as $\bar{K} = \text{blkdiag}(I_{n_x}, K)$, where K stabilizes P and n_x is the order of the plant. Since $\bar{P}$ is stable, all stabilizing controllers of congruent plant can be parametrized with a stable $\bar{Q}$ as $\bar{K} = -\bar{Q}(I - \bar{P}\bar{Q})^{-1}$ such that $\bar{K}$ is in the form of $\text{blkdiag}(I_{n_x}, K)$ where K stabilizes P. By regarding these, we parametrize all stabilizing controllers benefiting all stabilizing controller parametrization of stable plant and by introducing constraints on Youla parameter such that the controller of congruent plant yields a controller structured as $\text{blkdiag}(I_{n_x}, K)$, which equivalently brings in the controller of original plant as K. The formulated all stabilizing controller parametrization does not require to have an initial controller or doubly-coprime factorization of plant unlike the well-known all stabilizing controller parametrization.

All stabilizing controller parametrization has been formulated with equality constraints, instead of solving for a feasible solution for these equality constraints in finite dimensional space, we define a necessary and sufficient controller problem by subjecting these equality constraints to norm minimization to be able to define and solve the controller problem in infinite dimensional space. We further define an equivalent stabilization problem by benefiting orthogonal spaces where the number of variables and equations necessary to solve are reduced. Moreover, we provide a two-step solution procedure for unconstrained controller problem which can be solved as a classical $\mathcal{H}_2$ problem wherein the solution lies in infinite dimensional space.

By network implementable system, we mean a system with network implementable state-space realization which can be expressed as a strictly causal interaction of some sub-systems over the given network. Moreover, by network realizable system we mean a system whose network implementable state-space realization can be obtained. Obtained all stabilizing controller parametrization allows us also to attain doubly-coprime factorization of $\bar{K} = \text{blkdiag}(I_{n_x}, K)$ where K is a controller of the given plant such that when the controller inherits the delay and sparsity constraints of the graph, then coprime factors of $\bar{K}$ also inherits these sparsity and delay constraints. In literature, there exists a method to have network implementable state-space realization of a given stable structured system which inherits the network's sparsity and delay constraints of the graph [1]. Therefore, since coprime factors are stable by definition, they can be called as network realizable system when they inherit the sparsity and delay constraints of the given graph and their network implementable state space realizations can be obtained by the method introduced in [1]. Having network implementable state-space realization of coprimes of blkdiag(I_{n_x}, K) allows us to attain network implementable state-space realization of such structured controllers, i.e. controllers which inherit sparsity and delay constraints of the given network, as it will be shown in chapter 4. Capability to have network implementable state-space realization of such structured controllers allows us to define the network realizable controllers. Moreover, the formulated technique to attain network implementable state-space realization of network realizable controllers allows us to obtain network implementable controllers with fewer order with respect to the other realization methods as it will be shown in the section of numerical examples (see chapter 8).

By benefiting the formulated stabilization problem formulated in chapter 3, we formulate the necessary and sufficient network distributed controller problem in chapter 5 by introducing sparsity delay constraints on Youla parameter. By benefiting vectorization method as shown in [37], we define the equivalent unconstrained controller problem which can be solved using existing solution techniques to attain a solution in infinite dimensional space.

One of the main objective of this work is solving optimal network realizable controller problem for any networked system in infinite dimensional space. We provide a model matching problem affine in $\bar{Q}$ and define an optimal network realizable controller problem with necessary and sufficiency conditions benefiting the formulated all stabilizing network realizable controller parametrization. The provided optimal network realizable controller problem is a constrained problem, therefore, by benefiting a Lagrange multiplier and vectorization method, we provide an unconstrained optimal network realizable control problem which can be solved using existing solution methods. Our main difference from classical optimal control problems is that our problem formulation has been defined with necessary and sufficient conditions and does not require to have an initial stabilizing controller or doubly coprime factorization of plant.

In chapter 7, we provide an alternative way to parametrize all stabilizing network distributed controllers. We benefit from the all stabilizing controller parametrization defined for parallel systems which are controller and plant itself, since there exist a doubly coprime factorization of those parallel systems as shown in [40]. After parameterizing the all stabilizing controllers for parallel plants, one can obtain the all stabilizing controller parametrization for the plant by feedback interconnection of initial stabilizing controller and the all stabilizing controller parametrization for parallel systems [27]. Bezout identity elements that satisfy the doubly coprime factorization of parallel plants: initial stabilizing controller and plant, obey the network structure when the initial stabilizing controller is network realizable. Therefore, we are able to define the network realizable all stabilizing controllers using an initial network realizable controller and a network realizable Youla-parameter. We obtain a model matching problem to have input to output map affine in Youla parameter. By using this model matching problem, we define optimal network realizable controller problem as a function of a network realizable controller. Furthermore, we benefit from the vectorization technique shown in [37] to be able to solve the optimal controller problem as an unconstrained problem benefiting existing solution methods wherein solution lies in infinite dimensional space.

The outline of this thesis is as follows: Chapter 2 presents the problem formulation and introduces some notation which will be used throughout the paper. In chapter 3, an all stabilizing controller parametrization has been formulated, moreover, we formulate a necessary and sufficient controller problem wherein a solution can be obtained in infinite dimensional space. In chapter 4, a doubly-coprime factorization of controller has been introduced which allows to have network implementable state-space realization of network distributed controllers in reduced order which is one of the main contribution of this paper. Furthermore, in chapter 5, a necessary and sufficient network realizable controller problem has been formulated for any network distributed system as an unconstrained problem which can be solved using existing solution methods wherein solution lies in infinite dimensional space. Moreover, optimal network realizable controller problem has been formulated in chapter 6. In chapter 7, we provide an alternative way to parametrize all stabilizing network distributed controllers benefiting existing all stabilizing controller parametrization. Finally, the work ends with two numerical examples and comparisons with the existing optimal controller problems.

CHAPTER 2. PRELIMINARIES AND DEFINITIONS

In this chapter, we will give definitions that will be used throughout this work. Also, we will provide theorems that we will benefit to derive our results.

2.1 General Notation

In this section we will provide general notation that will be used throughout this work.

Let $\mathbf{vert}[x_i]_{i\in\mathbb{I}}$ and $\mathbf{hor}[x_i]_{i\in\mathbb{I}}$ denote for vertical and horizontal concatenation of vectors or matrices $\{x_i\}_{i\in\mathbb{I}}$ of appropriate dimension, where $\mathbb{I}$ is an index set. Let $[x_{ij}]_{i,j\in\mathbb{I}}$ represent a matrix formed by arranging the sub-matrices $\{x_{ij}\}_{i,j}$ as $\mathbf{vert}[\mathbf{hor}[x_{ij}]_{j\in\mathbb{I}}]_{i\in\mathbb{I}}$. Moreover, $\mathbf{diag}[x_i]_{i\in\mathbb{I}}$ denotes the block diagonal matrix formed by matrices $\{x_i\}_{i\in\mathcal{I}}$.

Feedback interconnection of P and K is represented with $\mathbf{lft}(P,K)$. Moreover, $B_u^\dagger$ and $C_y^\dagger$ implies Moore-Penrose inverse of B_u and C_y such that $B_uB_u^\dagger B_u = B_u$ and $C_yC_y^\dagger C_y = C_y$. When B_u has linearly independent columns, then $B_u^\dagger$ can be given as $B_u^\dagger = (B_u^TB_u)^{-1}B_u^T$. Moreover, when C_y has linearly independent rows than its pseudo inverse can be given as $C_y^\dagger = C_y^T(C_yC_y^T)^{-1}$. Furthermore, we use $\mathbf{blkdiag}(A_1, A_2)$ to define a block diagonal matrix constructed with its input arguments such that $\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$.

If $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$, the Kronecker product $A \otimes B \in \mathbb{R}^{mp\times nq}$ is defined as

$$A \otimes B := \begin{bmatrix} A_{11}B & \dots & A_{1n}B \\ \vdots & \ddots & \vdots \\ A_{m1}B & \dots & A_{mn}B \end{bmatrix}.$$

Given a matrix $A = \begin{bmatrix} a_1, ..., a_n \end{bmatrix} \in \mathbb{C}^{m \times n}$, where $\{a_i\}_i$ denote the columns of A, we associate a vector $\mathbf{vec}(A) = \mathbf{vert}\begin{bmatrix} a_i \end{bmatrix}_i \in \mathbb{C}^{mn}$ which is a vector formed by vertically concatenating the columns of matrix A. Define $\mathbf{vec}^{-1}(\cdot)$ as the inverse operation of the $\mathbf{vec}(\cdot)$ such that $\mathbf{vec}^{-1}(\mathbf{vec}(A)) = A$. When required, we shall use I for an identity matrix and 0 for a zero matrix of appropriate size.

2.2 System Theory

A system P is represented by a quadruple (A, B, C, D) or

$$P : \begin{bmatrix} x(k+1) \\ y(k) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}, \tag{2.1}$$

in terms of its state-space matrices A, B C and D; and state, input and output vectors $x(k)$, $u(k)$ and $y(k)$, respectively. A state-space representation (A, B, C, D) is asymptotically stable if A is Schur stable. (A, B, C, D) is said to be *stabilizable* if $\begin{bmatrix} zI - A & B \end{bmatrix}$ has full rank for any $z \in \mathbb{C}$ with $z \geq |1|$. (A, B, C, D) is said to be *detectable* if $\begin{bmatrix} zI - A \\ C \end{bmatrix}$ has full rank for any $z \in \mathbb{C}$ with $z \geq |1|$.

Given a state-space representation (A, B, C, D), the transfer function matrix corresponding to the system P is given by the z–transform of its impulse response

$$P(z) := \mathbf{tf}(P) := D + \sum_{k=0}^{\infty} z^{-k-1} C A^k B. \tag{2.2}$$

For given two systems G and K in terms of their state-space representations

$$G : \begin{bmatrix} x(k+1) \\ z(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}, \quad K : \begin{bmatrix} x_K(k+1) \\ u(k) \end{bmatrix} = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} \begin{bmatrix} x_K(k) \\ y(k) \end{bmatrix}, \tag{2.3}$$

the lower linear fractional transformation (LFT) of G and K is given by the Redheffer star-product

$$\mathbf{lft}(G,K):\begin{bmatrix} x(k+1) \\ x_K(k) \\ z(k) \end{bmatrix} = \begin{bmatrix} A+B_2D_KC_2 & B_2C_K & B_1+B_2D_KD_{21} \\ B_KC_2 & A_K & B_KD_{21} \\ C_1+D_{12}D_KC_2 & D_{12}C_K & D_{11}+D_{12}D_KD_{21} \end{bmatrix} \begin{bmatrix} x(k) \\ x_K(k) \\ w(k) \end{bmatrix}. \tag{2.4}$$

When these two systems are given in terms of their transfer function matrices $G(z)$ and $K(z)$ where $G(z)$ is the mapping from $\begin{bmatrix} w(k) \\ u(k) \end{bmatrix}$ to $\begin{bmatrix} z(k) \\ y(k) \end{bmatrix}$ while $K(z)$ is the mapping from $y(k)$ to $u(k)$, we can partition the transfer function matrix $G(z)$ in terms $G_{11}(k)$, $G_{12}(z)$, $G_{21}(z)$ and $G_{22}(z)$ as

$$G(z) := \begin{bmatrix} G_{11}(z) & G_{12}(z) \\ G_{21}(z) & G_{22}(z) \end{bmatrix},$$

where $G_{22}(z)$ is the mapping from $u(k)$ to $y(k)$. Then the LFT of $G(z)$ and $K(z)$ is given by

$$\mathbf{lft}(G(z),K(z)) := G_{11}(z)+G_{12}(z)K(z)(I-G_{22}(z)K(z))^{-1}G_{21}(z).$$

when D matrix of G_{22} is zero, i.e. $G_{22}(z)$ is strictly proper.

A discrete-time system is called as *bounded-input bounded-output (BIBO) stable*, if the impulse response of the system is absolutely summable. A system G is BIBO stable if and only if all the poles of its transfer function matrix $G(z)$ are inside the unit circle. A discrete-time system G with a state-space representation (A,B,C,D) is called as *internally stable* or *asymptotically stable* if A is Schur-stable. If $G=(A,B,C,D)$ is asymptotically stable, then $\mathbf{tf}(G)$ is BIBO stable, but not vice versa. We say that a system K is a controller of G or K *internally stabilizes* G if $\mathbf{lft}(G,K)$ is asymptotically stable.

For a given discrete-time system G, $\mathcal{H}_2$ norm of the system can be given as

$$||G(z)||_2^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbf{tr}(G(e^{j\theta})G^*(e^{j\theta})d\theta)$$

wherein $G(z)$ is the transfer function matrix of G. If a state-space realization of G is given by (A, B, C, D), then $\mathcal{H}_2$ norm of G can be given as

$$||G||_2^2 = \mathbf{tr}(DD^T + CM_cC^T),$$

where $M_c \succeq 0$ is the controllability grammian that solves the discrete-time Lyapunov equation

$$AM_cA^T - M_c + BB^T = 0. \tag{2.5}$$

Moreover, solution of equation (2.5) can be given as

$$M_c = \sum_{k=0}^{\infty} A^k BB^T (A^T)^k.$$

Moreover, $\mathcal{RH}_\infty$ denotes the set of real-rational proper stable transfer function matrices.

2.3 Network Distributed Systems

In this section, we will introduce network distributed systems and will provide a general representation of them.

Definition 1. *A group of subsystems interacting over a communication network is defined as a networked or distributed or an interconnected system. [1]*

Figure 2.1 is an example of a network distributed system, which consists of network distributed plants and their controller units.

Consider n sub-systems $\{P_i\}_{i\in\{1,\dots,n\}}$ interacting over a network represented by directed pseudograph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with the sub-systems at its vertices and communication links corresponding to the edges. Edge set can be defined as $(i, j) \in \mathcal{E}$ which denotes that there is an edge between P_i and P_j. Directed neighborhood index sets for each node P_i are $\mathcal{N}_i^- = \{j|(j, i) \in \mathcal{E}\}$, $\mathcal{N}_i^+ = \{j|(i, j) \in \mathcal{E}\}$ and $\mathcal{N}_i = \{j|(j, i) \in \mathcal{E} \vee (i, j) \in \mathcal{E}\} = \mathcal{N}_i^- \cup \mathcal{N}_i^+$, where $\mathcal{N}_i^-$ and $\mathcal{N}_i^+$ are incoming and outgoing neighbor sets of node P_i, respectively.

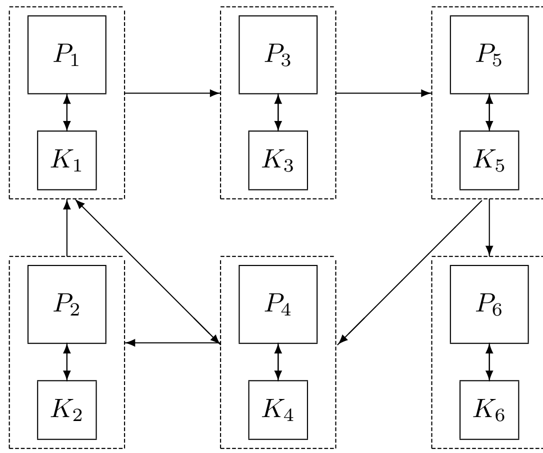

Figure 2.1 An example of a distributed system which consists of 6 sub-systems and their controller units interacting over a causal network.

For a given graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, the unique binary matrices can be defined as in the following

$$[\mathcal{A}(\mathcal{G})]_{ij} = \begin{cases} 1 & \text{if } i = j \text{ or } (i,j) \in \mathcal{E} \\ 0 & \text{else.} \end{cases} \tag{2.6}$$

We use $\mathcal{G}^n$ to define $\mathbf{1}_n \otimes \mathcal{G}$, for instance $\mathcal{G}^2 = \begin{bmatrix} \mathcal{G} & \mathcal{G} \\ \mathcal{G} & \mathcal{G} \end{bmatrix}$.

Definition 2. *A directed graph is strongly connected if there is a path from every node to every other node.*

In this work, we will come across with block matrices that are made up of smaller sub-matrices. These matrices are best described in terms of their sparsity structures. We say a block matrix $A = [A_{ij}]_{i,j \in \{1,\dots,n\}}$ is structured according to an $n \times n$ binary matrix J if the sub-matrices A_{ij} is a zero matrix whenever $J_{ij} = 0$. The dimensions of the sub-matrices $\{A_{ij}\}_{i,j}$ are described using two integer-valued vectors as follows. Let $\mathcal{P}_a = (a_1, \dots, a_n)$ and $\mathcal{P}_b = (b_1, \dots, b_n)$ be two n-tuples with a_i and b_i being integers for all $i \in \{1, \dots, n\}$. Then, matrix A is said to be partitioned according to $(\mathcal{P}_a, \mathcal{P}_b)$ if the sub-matrix A_{ij} has dimensions $a_i \times b_j$ $\forall i, j$. This definition of partitioning can be easily extended to the case of vectors, too.

A vector x is said to be partitioned according to $\mathcal{P}_a$ if it can be written as $\mathbf{vert}[x_i]_{i\in\{1,\ldots,n\}}$ where x_i is a real vector of size a_i for all $i \in \{1, ..., n\}$. We say that $\mathcal{P}_a$ is the partition for the vector x.

Definition 3. *Given an $n \times n$ binary matrix J and n–tuples $\mathcal{P}_a$, $\mathcal{P}_b$, $S(J, \mathcal{P}_a, \mathcal{P}_b)$ denotes the set of matrices that are partitioned according to $(\mathcal{P}_a, \mathcal{P}_b)$ and structured according to J.*

For example, according to the above definition, the following matrix

$$A = \left[\begin{array}{c|cc|ccc} 1 & 2 & 3 & 0 & 0 & 0 \\ \hline 2 & 1 & 1 & 0 & 0 & 0 \\ 3 & 1 & 2 & 0 & 0 & 0 \\ \hline 0 & 1 & 3 & 2 & 2 & 1 \end{array}\right] \tag{2.7}$$

belongs to set $A \in S(J, \mathcal{P}_a, \mathcal{P}_b)$ wherein $J = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, $\mathcal{P}_a = (1, 2, 1)$ and $\mathcal{P}_b = (1, 2, 3)$.

Let each subsystem P_i be a discrete-time causal finite-dimensional linear time invariant system. State space equations of each stable subsystem P_i can be given as

$$\begin{aligned} x_i(k+1) &= A_{ii}x_i(k) + B_i^w w_i(k) + B_i^u u_i(k) + \sum_{j\in\mathcal{N}_i^-} B_{ij}^{\zeta}\zeta_{ij}(k) \\ z_i(k) &= C_{ii}^z x_i(k) + D_i^{zw} w_i(k) + D_i^{zu} u_i(k) + \sum_{j\in\mathcal{N}_i^-} D_{ij}^{z\zeta}\zeta_{ij}(k) \\ y_i(k) &= C_{ii}^y x_i(k) + D_i^{yw} w_i(k) + \sum_{j\in\mathcal{N}_i^-} D_{ij}^{y\zeta}\zeta_{ij}(k) \\ \eta_{ri} &= C_{ri}^{\eta} x_i(k), \ \forall r \in \mathcal{N}_i^+ \end{aligned} \tag{2.8}$$

where $x_i(k)$, $w_i(k)$, $u_i(k)$, $z_i(k)$ and $y_i(k)$, $\eta_{ri}(k)$, and $\zeta_{ij}(k)$ denote the local state, local exogenous input, local control input, local regulated output, local measurement output vectors, local outputs to the network, and local inputs from the network corresponding to a networked subsystem P_i, respectively. For a given network $\mathcal{G}$, incoming message vectors at each node are given by

$$\zeta_{ij}(k) = \eta_{ij}(k), \ \forall (v_j, v_i) \in \mathcal{E} \tag{2.9}$$

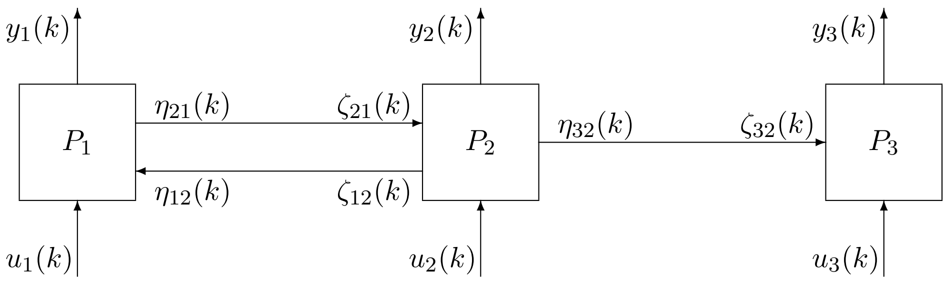

Figure 2.2 A simple example of an interconnected system made of 3 different sub-systems.

Definition 4. *Networked systems given by* (2.8) *and* (2.9) *are denoted as strictly causal interaction of subsystems over a given network* $\mathcal{G}$.

Combining (2.8) and (2.9), network inputs and outputs can be eliminated to have the state-space equations for the subsystems as

$$\begin{aligned} x_i(k+1) &= A_{ii}x_i(k) + \sum_{j\in\mathcal{N}_i^-} A_{ij}x_j(k) + B_i^w w_i(k) + B_i^u u_i(k) \\ z_i(k) &= C_{ii}^z x_i(k) + \sum_{j\in\mathcal{N}_i^-} C_{ij}^z x_j(k) + D_i^{zw} w_i(k) + D_i^{zu} u_i(k) \\ y_i(k) &= C_{ii}^y x_i(k) + \sum_{j\in\mathcal{N}_i^-} C_{ij}^y x_j(k) + D_i^{yw} w_i(k) \end{aligned} \tag{2.10}$$

where $A_{ij} := B_{ij}^{\zeta} C_{ij}^{\eta}$, $C_{ij}^z := D_{ij}^{z\zeta} C_{ij}^{\eta}$ and $C_{ij}^y := D_{ij}^{y\zeta} C_{ij}^{\eta}$. General networked system P can be given as

$$P : \begin{bmatrix} x(k+1) \\ z(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \\ u(k) \end{bmatrix} \tag{2.11}$$

where $A := [A_{ij}]_{i,j}$, $C_z := [C_{ij}^z]_{i,j}$ and $C_y := [C_{ij}^y]_{i,j}$ are structured according to $\mathcal{A}(\mathcal{G})$, while $B_w := \mathbf{diag}[B_{ii}^w]_i$, $B_u := \mathbf{diag}[B_{ii}^u]_i$, $D_{zw} := \mathbf{diag}[D_{ii}^{zw}]_i$, $D_{zu} := \mathbf{diag}[D_{ii}^{zu}]_i$ and $D_{yw} := \mathbf{diag}[D_{ii}^{yw}]_i$ have a block diagonal structure.

For the given interconnected system in figure 2.2 consists of 3 sub-systems, $A \in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_x, \mathcal{P}_x)$, $C_y \in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_y, \mathcal{P}_x)$ and $C_z \in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_z, \mathcal{P}_x)$ matrices can be given as follows

$$A = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & A_{32} & A_{33} \end{bmatrix}, \quad C_{z,y} = \begin{bmatrix} C_{11}^{z,y} & C_{12}^{z,y} & 0 \\ C_{21}^{z,y} & C_{22}^{z,y} & 0 \\ 0 & C_{32}^{z,y} & C_{33}^{z,y} \end{bmatrix}, \tag{2.12}$$

Moreover, $B_w \in S(I, \mathcal{P}_x, \mathcal{P}_w)$, $B_u \in S(I, \mathcal{P}_x, \mathcal{P}_u)$, $D_{zw} \in S(I, \mathcal{P}_z, \mathcal{P}_w)$, $D_{zu} \in S(I, \mathcal{P}_z, \mathcal{P}_u)$ and $D_{yw} \in S(I, \mathcal{P}_y, \mathcal{P}_w)$ matrices can be given as follows

$$B_{w,u} = \begin{bmatrix} B_1^{w,u} & 0 & 0 \\ 0 & B_2^{w,u} & 0 \\ 0 & 0 & B_3^{w,u} \end{bmatrix}, \quad D_{zw,zu,yw} = \begin{bmatrix} D_1^{zw,zu,yw} & 0 & 0 \\ 0 & D_2^{zw,zu,yw} & 0 \\ 0 & 0 & D_3^{zw,zu,yw} \end{bmatrix}. \tag{2.13}$$

Definition 5. *[37] Given a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with n vertices and n tuples $\mathcal{P}_x$, $\mathcal{P}_u$ and $\mathcal{P}_y$; let $\mathcal{A}(\mathcal{G})$ be the unique binary matrix as defined in equation (2.6). We define $\mathfrak{T}(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$ as the set of transfer function matrices, and $\mathfrak{S}(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_y, \mathcal{P}_u)$ as the set of state-spaces with a state-space realization (A, B_u, C_y, D_{yu}) such that $A \in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_x, \mathcal{P}_x)$, $B_u \in S(I, \mathcal{P}_x, \mathcal{P}_u)$, $C_y \in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_y, \mathcal{P}_x)$ and $D_{yu} \in S(I, \mathcal{P}_y, \mathcal{P}_u)$, where I is identity matrix and sets belong to $S(I, \cdot, \cdot)$ are block diagonal.*

For $A \in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_x, \mathcal{P}_x)$, $B_u \in S(I, \mathcal{P}_x, \mathcal{P}_u)$, $C_y \in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_y, \mathcal{P}_x)$ matrices given as in (2.12) and (2.13) and $D_{yu} \in S(I, \mathcal{P}_y, \mathcal{P}_u)$, plant $P_{22} = \mathbf{ss}(A, B_u, C_y, D_{yu}) \in \mathfrak{S}(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_y, \mathcal{P}_u)$ has the following structure in z-domain

$$P_{22}(z) = \begin{bmatrix} P_{11}^{22}(z) & z^{-1}P_{12}^{22}(z) & 0 \\ z^{-1}P_{21}^{22}(z) & P_{22}^{22}(z) & 0 \\ z^{-2}P_{31}^{22}(z) & z^{-1}P_{32}^{22}(z) & P_{33}^{22}(z) \end{bmatrix}, \tag{2.14}$$

where P_{ij}^{22} for $\{i, j\} \in \{1, 2, 3\}$ are casual systems and we have $P_{22}(z) \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$. As it can be noticed from $P_{22}(z)$, since the length of path from node-1 to node-2 is one, we observe one delay in front of $P_{21}^{22}(z)$, similarly since the length of path from node-1 to node-3 is two, we observe two delays in front of $P_{31}^{22}(z)$ and so on. Moreover, since there is no directed path from node-3 to node-1 and node-2, we have zero entries on the places of $P_{13}^{22}(z)$ and $P_{23}^{22}(z)$.

Theorem 1. *[37, Theorem 1] Given a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and n tuple $\mathcal{P}_u$ and $\mathcal{P}_y$.*

1. *Let $P(z)$ be a transfer function matrix in $\mathfrak{T}(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$ with input vector $u(k)$ and output vector $y(k)$ partitioned according to $\mathcal{P}_u$, $\mathcal{P}_y$, respectively. Then there exists a state-space realization (A, B_u, C_y, D_{yu}) of $P(z)$ in $\mathfrak{S}(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_y, \mathcal{P}_u)$ with state vector $x(k)$ partitioned according to some n tuple $\mathcal{P}_x$.*

2. *If $P(z)$ is also BIBO stable, i.e. $P(z) \in \mathfrak{T}^s(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$, then there exists a state-space realization (A, B_u, C_y, D_{yu}) of $P(z)$ in $\mathfrak{S}^s(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_y, \mathcal{P}_u)$ for some n tuple P_x, i.e. A is Schur-stable.*

Definition 6. *[1] We refer to the property to realizing a structured transfer function matrix in $\mathfrak{T}(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$ as a stabilizable and detectable networked system which is a strictly causal interaction over $\mathcal{G}$ with the same transfer function as network realizability.*

Definition 7. *Stabilizable and detectable system's state space realization $P \in \mathfrak{S}(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_y, \mathcal{P}_u)$ can be implemented as strictly causal interaction of subsystems (2.8) and (2.9) over a given graph $\mathcal{G}$. Such systems are said to be network implementable. Moreover, such state-space realizations are said to be network implementable state-space realization.*

Theorem 1 ensures that a stable system P with transfer function matrix $P(z) \in \mathfrak{T}^s(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$ is *network realizable* over $\mathcal{G}$ with a *network implementable state-space realization* $\tilde{P} \in \mathfrak{S}(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_y, \mathcal{P}_u)$ such that $P(s) = \mathbf{tf}(\tilde{P})$, where '**tf**' is the operator transforms state-space system into transfer function.

The sets of asymptotically stable structured systems over the network interconnection $\mathcal{G}$ with input and output partitions as $\mathcal{P}_u$ and $\mathcal{P}_y$ are denoted by $\mathfrak{S}^s(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_y, \mathcal{P}_u)$ and $\mathfrak{T}^s(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$, respectively.

2.4 Youla Parametrization

Herein, we state a definition of doubly coprime factorization which will be used to define all stabilizing controller parametrization, then we will review well known Youla-Kučera all stabilizing controller parametrization.

Definition 8. *[5] A doubly coprime factorization of* P_{22} *is a set of maps* N, M, $\tilde{N}$, $\tilde{M}$, *with* $\bar{P}_{22} = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ *satisfying*

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & Y \\ N & X \end{bmatrix} = I, \tag{2.15}$$

for some stable X, Y, $\tilde{X}$ *and* $\tilde{Y}$. *Further,* M *and* N *are referred to as right coprime factors while* $\tilde{M}$ *and* $\tilde{N}$ *are referred to as left coprime factors of* P_{22}.

Theorem 2. *[5] Let a doubly coprime factorization of* P_{22} *be given as in definition 8. All stabilizing controllers of* P_{22} *can be given as*

$$K = (Y - MQ)(X - NQ)^{-1} = (\tilde{X} - Q\tilde{N})^{-1}(\tilde{Y} - Q\tilde{M}) \tag{2.16}$$

with $Q \in \mathcal{RH}_\infty$.

Parametrization in (2.16) allows one to define all possible controllers when a doubly coprime factorization of plant is given.

CHAPTER 3. AN ALTERNATIVE CHARACTERIZATION OF STABILIZATION

In this chapter, we first derive all stabilizing controller parametrization of stable plants. We define internally stabilizing controllers by re-interpreting plant and controller. We define a congruent stable plant of the original plant which is not necessarily stable, such that controller of the congruent plant is linearly function of the original plant's controller. When we put structural constraints on the all stabilizing controllers of the stable congruent plant, these controllers embody the controllers of the main plant. Therefore, all stabilizing controller problem of the original plant is defined as all stabilizing controllers of the congruent stable plant with structural constraints. Regarding this problem, we obtain all stabilizing controller parametrization of any plant benefiting all stabilizing controller parametrization of stable plants. All stabilizing controller parametrization is obtained with equality constraints wherein Youla parameter is the variable, instead of solving for a feasible solution of these constraints, we subject these equality constraints to norm minimization to define a necessary and sufficient controller problem. We further reduce the number of variables in stabilization problem benefiting orthogonal spaces and provide a two-step procedure to solve and obtain a solution in infinite dimensional space. Moreover, we provide necessary and sufficient stabilizability and detectability test problems, then we formulate necessary and sufficient problems to have a dynamic state feedback controller and state observer benefiting all stabilizing controller parametrization of stable plants.

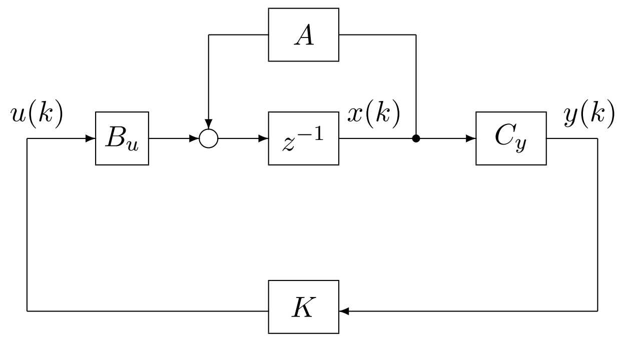

Figure 3.1 A block diagram of feedback interconnection of a plant and its controller, $\mathbf{lft}(P_{22}, K)$.

3.1 All Stabilizing Controller Parametrization

In this section, we will first derive all stabilizing controller parametrization of stable plants, then we will define a stably defined congruent plant of given any stable/unstable plant and its controller to benefit from all stabilizing controller parametrization of stable plants. Afterwards, we derive all stabilizing controller parametrization wherein equality constraints are introduced on Youla parameter by benefiting all stabilizing controller parameterization of stable plants.

3.1.1 Problem Formulation

In the next lemma we provide all stabilizing controller parametrization of stable plants.

Lemma 1. *All stabilizing controllers of stable plant P_{22} can be parametrized as $K = -Q(I - P_{22}Q)^{-1}$ with $Q \in \mathcal{RH}_\infty$.*

Proof of lemma 1 can be found at appendix A.1.

Let the generalized plant be defined as in the following

$$P : \begin{bmatrix} x(k+1) \\ z(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \\ u(k) \end{bmatrix} \tag{3.1}$$

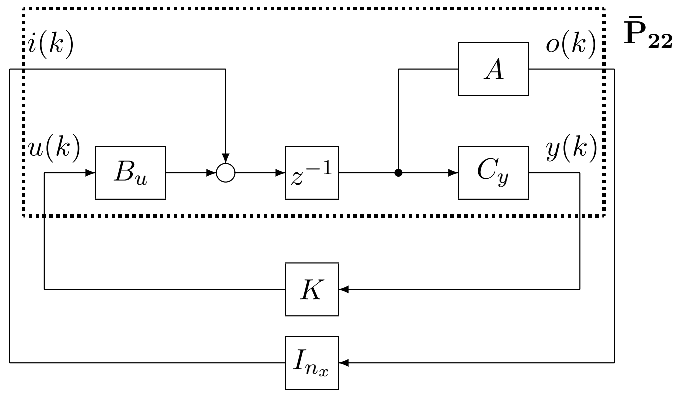

Figure 3.2 An equivalent block diagram of feedback interconnection of a plant and its controller, $\mathbf{lft}(\bar{P}_{22}, \mathbf{blkdiag}(I_{n_x}, K))$.

According to lemma 1, when given plant P_{22} is stable, all stabilizing controllers of P_{22} can be parametrized as

$$K = -Q(I - P_{22}Q)^{-1} \tag{3.2}$$

with a stable Q. However, when given plant is unstable, one can not benefit from this all stabilizing controller parametrization. Hence, we define a stable congruent plant of the original plant in larger dimensions to be able to benefit the controller parametrization given in (3.2).

As it can be trivially observed from figure 3.1 and figure 3.2, block diagram exists in figure 3.1 is equivalent to block diagram exists in figure 3.2. Regarding figure 3.2, we can define the feedback interconnection with two systems, stable congruent plant, $\bar{P}_{22}$, and its controller, $\bar{K}$, which are

$$\bar{P}_{22} = \begin{bmatrix} z^{-1}A & z^{-1}AB_u \\ z^{-1}C_y & z^{-1}C_yB_u \end{bmatrix}, \tag{3.3a}$$

$$\bar{K} = \begin{bmatrix} I_{n_x} & 0 \\ 0 & K \end{bmatrix}. \tag{3.3b}$$

Then, feedback interconnection of P_{22} and controller K is equivalent to feedback interconnection of $\bar{P}_{22}$ and $\bar{K}$. As it can also be observed from block diagram of $\bar{P}_{22}$ in figure 3.2, when input channel-i and output channel-o gets connected with a unity feedback, input u to output y map becomes equivalent to P_{22}.

Since $\bar{P}_{22}$ given in (3.3a) is stable, one can parametrize all stabilizing controllers of $\bar{P}_{22}$ with a stable $\bar{Q}$ as $\bar{K} = -\bar{Q}(I - \bar{P}_{22}\bar{Q})^{-1}$. Moreover, if $\bar{K}$ is structured as $\bar{K} = \textbf{blkdiag}(I_{n_x}, K)$, then K is a controller of P_{22}.

3.1.2 All Stabilizing Controller Parametrization

Next, we define the constraints need to be satisfied by $\bar{Q}$ to be able to obtain all stabilizing controllers of $\bar{P}_{22}$ structured as $\bar{K} = \textbf{blkdiag}(I_{n_x}, K)$ by all stabilizing controller parametrization $\bar{K} = -\bar{Q}(I - \bar{P}_{22}\bar{Q})^{-1}$, which will equivalently bring us all stabilizing controllers of P_{22}.

Lemma 2. *Let P be defined as in (2.11) and let $\bar{Q} := \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix}$. All stabilizing controllers of P can be parametrized as $K = -Q_4(I - P_{22}Q_4)^{-1}$ where $\bar{Q} \in \mathcal{RH}_\infty$ satisfies the followings*

$$\begin{bmatrix} z^{-1}A - I & z^{-1}AB_u \end{bmatrix} \bar{Q} = \begin{bmatrix} I & 0 \end{bmatrix}, \tag{3.4a}$$

$$\bar{Q} \begin{bmatrix} z^{-1}A - I \\ z^{-1}C_y \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}. \tag{3.4b}$$

Proof. By regarding lemma 1, since $\bar{P}_{22}$ given in (3.3) is stable, all stabilizing controllers of $\bar{P}_{22}$ can be parametrized as in the following

$$\bar{K} = -\bar{Q}(I - \bar{P}_{22}\bar{Q})^{-1} \tag{3.5}$$

with $\bar{Q} \in \mathcal{RH}_\infty$. In order to have a stabilizing K also for the given plant P, we need to find a $\bar{Q}$ which induces a structured $\bar{K} := \textbf{blkdiag}(I_{n_x}, K)$ such that $\textbf{lft}(\bar{P}_{22}, \bar{K})$ is stable. By multiplying equation (3.5) from right with $(I - \bar{P}_{22}\bar{Q})$ and using the definition of $\bar{K}$, we obtain the followings

$$\bar{K}(I - \bar{P}_{22}\bar{Q}) = -\bar{Q}$$

$$\begin{bmatrix} I & 0 \\ 0 & K \end{bmatrix} \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} z^{-1}A & z^{-1}AB_u \\ z^{-1}C_y & z^{-1}C_yB_u \end{bmatrix} \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix} \right) = - \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix} \quad (3.6)$$

$$\begin{bmatrix} I - z^{-1}AQ_1 - z^{-1}AB_uQ_3 & -z^{-1}AQ_2 - z^{-1}AB_uQ_4 \\ K(-z^{-1}C_yQ_1 - z^{-1}C_yB_uQ_3) & K(I - z^{-1}C_yQ_2 - z^{-1}C_yB_uQ_4) \end{bmatrix} = \begin{bmatrix} -Q_1 & -Q_2 \\ -Q_3 & -Q_4 \end{bmatrix}$$

Using last matrix equality, we obtain the following equations

$$(z^{-1}A - I)Q_1 + z^{-1}AB_uQ_3 = I, \quad (3.7a)$$

$$(z^{-1}A - I)Q_2 + z^{-1}AB_uQ_4 = 0, \quad (3.7b)$$

$$K(-z^{-1}C_yQ_1 - z^{-1}C_yB_uQ_3) = -Q_3, \quad (3.7c)$$

$$K(I - z^{-1}C_yQ_2 - z^{-1}C_yB_uQ_4) = -Q_4. \quad (3.7d)$$

Using (3.7b) we can express Q_2 as follows

$$Q_2 = (zI - A)^{-1}AB_uQ_4 \quad (3.8)$$

By plugging definition of Q_2 as in (3.8) into (3.7d), we obtain the followings

$$\begin{aligned} K(I - z^{-1}C_y(zI - A)AB_uQ_4 - z^{-1}C_yB_uQ_4) &= -Q_4, \\ K(I - P_{22}Q_4) &= -Q_4. \end{aligned} \quad (3.9)$$

By multiplying (3.9) from right with $(I - P_{22}Q_4)^{-1}$, we obtain the following

$$K = -Q_4(I - P_{22}Q_4)^{-1}. \quad (3.10)$$

Moreover, one can equivalently parametrize $\bar{K}$ as $\bar{K} = -(I - \bar{Q}\bar{P}_{22})^{-1}\bar{Q}$, using this parametrization we obtain the followings

$$
\begin{gathered}
(I - \bar{Q}\bar{P}_{22})\bar{K} = -\bar{Q} \\
\left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix} \begin{bmatrix} z^{-1}A & z^{-1}AB_u \\ z^{-1}C_y & z^{-1}C_yB_u \end{bmatrix} \right) \begin{bmatrix} I & 0 \\ 0 & K \end{bmatrix} = - \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix} \\
\begin{bmatrix} I - z^{-1}Q_1A - z^{-1}Q_2C_y & -z^{-1}Q_1AB_uK - z^{-1}Q_2C_yB_uK \\ -z^{-1}Q_3A - z^{-1}Q_4C_y & K - z^{-1}Q_3AB_uK - z^{-1}Q_4C_yB_uK \end{bmatrix} = \begin{bmatrix} -Q_1 & -Q_2 \\ -Q_3 & -Q_4 \end{bmatrix}
\end{gathered} \tag{3.11}
$$

Using last matrix equality, we obtain the following equations

$$Q_1(z^{-1}A - I) + z^{-1}Q_2C_y = I, \tag{3.12a}$$

$$-z^{-1}Q_1AB_uK - z^{-1}Q_2C_yB_uK = -Q_2, \tag{3.12b}$$

$$Q_3(z^{-1}A - I) + z^{-1}Q_4C_y = 0, \tag{3.12c}$$

$$K - z^{-1}Q_3AB_uK - z^{-1}Q_4C_yB_uK = -Q_4. \tag{3.12d}$$

One can obtain equations (3.7c), (3.12b) and (3.12d) using (3.7a), (3.7b), (3.12a), (3.12c) and (3.10). Therefore, equations (3.7a), (3.7b), (3.12a) and (3.12c) constitutes sufficient equations to solve for a solution. Equation (3.6) inherits the structural property of $\bar{K}$ such that $\bar{K} = \mathbf{blkdiag}(I_{n_x}, K)$. Hence, for a $\bar{Q}$ which satisfies the equations in (3.7a), (3.7b), (3.12a) and (3.12c), we can obtain K as $K = -Q_4(I - P_{22}Q_4)^{-1}$ and construct $\bar{K}$ as $\bar{K} = \mathbf{blkdiag}(I_{n_x}, K)$ which is a stabilizing controller of $\bar{P}_{22}$, i.e. $\mathbf{lft}(\bar{P}_{22}, \bar{K})$ is stable, equivalently we have $\mathbf{lft}(P_{22}, K)$ is stable. □

Lemma 2 allows one to parametrize all stabilizing controllers of P_{22} without necessitating to have a doubly coprime factorization of P_{22} unlike well-known Youla-Kucera all stabilizing controller parametrization given in (2.16). There exists also all stabilizing controller parametrization developed in [43] which does not also require any priori computations like doubly coprime factorization of plant. Our main purpose of deriving this all stabilizing controller parametrization is to show that one can obtain all stabilizing controllers of the given plant by all stabilizing controller parametrization of a stably defined congruent stable plant with some structural constraints on its controller.

Next, we show that all stabilizing controller parametrization given in lemma 2 can be written with reduced number of equations.

Corollary 1. *Let P be defined as in (2.11). All stabilizing controllers of P can be parametrized as $K = -Q_4(I - P_{22}Q_4)^{-1}$ where $Q_4 \in \mathcal{RH}_\infty$ satisfies the followings*

$$\begin{bmatrix} z^{-1}A - I & z^{-1}AB_u \end{bmatrix} \bar{Q} = \begin{bmatrix} I & 0 \end{bmatrix}, \tag{3.13a}$$

$$Q_3(z^{-1}A - I) + z^{-1}Q_4C_y = 0. \tag{3.13b}$$

Proof of corollary 1 can be found at appendix A.2.

Corollary 1 allows us to parametrize all stabilizing controllers with reduced number of equality constraints.

3.1.3 All Stabilizing Controllers for the case of $D_{yu} \neq 0$

Let $K = \mathbf{ss}(A_K, B_K, C_K, D_K)$ be a controller of $P_{22} = \mathbf{ss}(A, B_u, C_y, D_{yu})$. In the case of $D_{yu} \neq 0$, one should have $I - D_K D_{yu}$ such that it is invertible to have a well-posed feedback interconnection. Moreover, to design the all controllers for the case of $D_{yu} \neq 0$, one can choose $\bar{P}_{22}$ as $\bar{P}_{22} = \begin{bmatrix} z^{-1}A & z^{-1}AB_u \\ z^{-1}C_y & z^{-1}C_yB_u + D_{yu} \end{bmatrix}$ and follow the steps in proof of lemma 2. Repeating the steps in proof of lemma 2, one will trivially come up with the same constraint set given in (3.4) to parametrize all stabilizing controllers of P_{22}. Moreover, using controller parametrization $K = -(I - Q_4P_{22})^{-1}Q_4$, we obtain that $D_K = -(I - D_{Q_4}D_{yu})^{-1}D_{Q_4}$ wherein D_{Q_4} is D matrix of system Q_4, therefore we need $I - D_{Q_4}D_{yu}$ as invertible to have a well-defined controller. Therefore, one need to impose constraint of $I - D_{Q_4}D_{yu}$ is invertible in addition to constraints given in (3.4) to parametrize all stabilizing controllers of P_{22} when $D_{yu} \neq 0$. Using $D_K = -(I - D_{Q_4}D_{yu})^{-1}D_{Q_4}$, we obtain $I - D_K D_{yu}$ as $(I - D_{Q_4}D_{yu})^{-1}$ whose inverse is $I - D_{Q_4}D_{yu}$ which shows that feedback interconnection of plant and controller is well-posed with this controller parametrization.

3.2 Output Feedback Controller Problem

As it is shown in section 3.1, one can parametrize all stabilizing controllers of P with a stable $\bar{Q}$ satisfying the constraint set (3.4). In this section, we will show that equation set in (3.4) can be reduced to one equation. One may solve equality constraint in finite dimensional space by defining finite impulse responses of the systems with not know a priory finite support solution. Instead of solving zero equality problem in finite dimensional space, we define the infinite dimensional problem by considering its norm minimization. The following theorem sets a necessary and sufficient problem to obtain an output feedback controller benefiting all stabilizing controller parametrization obtained in lemma 2.

Corollary 2. *Let the plant be as given in* (2.11)*. There exists an internally stabilizing controller, if and only if there exist* $\tilde{Q}_1$ *and* Q_4 *which make the objective of following problem zero.*

$$\begin{aligned} &\min_{\tilde{Q}_1, Q_4} \left\| A^2(z^{-1}A - I) - (z^{-1}A - I)\tilde{Q}_1(z^{-1}A - I) + AB_uQ_4C_y \right\|_2^2 \\ &s.t. \quad \tilde{Q}_1 \in \mathcal{RH}_\infty, Q_4 \in \mathcal{RH}_\infty. \end{aligned} \tag{3.14}$$

Moreover, let $\tilde{Q}_1^*$ *and* Q_4^* *be solution of* (3.14) *such that its objective is zero, then a controller can be constructed as* $K = -Q_4^*(I - P_{22}Q_4^*)^{-1}$.

Proof. Since equations given in (3.4) are necessary and sufficient constraints to have a controller according to lemma 2, they need to be satisfied with a feasible stable $\bar{Q}$ if there exists any controller. Using (3.12c) and (3.7b), we can write Q_2 and Q_3 in terms of Q_4 as follows

$$Q_2 = -z^{-1}(z^{-1}A - I)^{-1}AB_uQ_4, \tag{3.15a}$$

$$Q_3 = -z^{-1}Q_4C_y(z^{-1}A - I)^{-1}. \tag{3.15b}$$

Using definition of Q_2 given in (3.15a), we can write equivalent of equation (3.12a) as follows

$$(z^{-1}A - I)Q_1(z^{-1}A - I) = z^{-1}A - I + z^{-2}AB_uQ_4C_y. \tag{3.16}$$

Moreover, we can also write equivalent of equation (3.7a) as in (3.16), using definition of Q_3 given in (3.15b). Also, a stable Q_4 satisfying (3.16) ensures the stability of Q_2 and Q_3 by regarding equalities (3.7b) and (3.12c). Therefore, equations given in (3.4) can be reduced to one equation given in (3.16).

A Q_1 satisfying (3.16) admits a form $Q_1 = -I - z^{-1}A + z^{-2}\tilde{Q}_1$ where $\tilde{Q}_1$ is casual, therefore we can simplify (3.16) as follows

$$A^2(z^{-1}A - I) - (z^{-1}A - I)\tilde{Q}_1(z^{-1}A - I) + AB_uQ_4C_y = 0. \tag{3.17}$$

An equivalent constraint of (3.17) can be written as its norm is equal to zero as in the following equation

$$\left\| A^2(z^{-1}A - I) - (z^{-1}A - I)\tilde{Q}_1(z^{-1}A - I) + AB_uQ_4C_y \right\|_2^2 = 0. \tag{3.18}$$

Therefore, if there exists a solution of (3.14) such that its objective is zero, then a controller can be constructed as $K = -Q_4(I - P_{22}Q_4)^{-1}$ by regarding lemma 2. □

Problem (3.14) allows us to define a stabilization problem in infinite dimensional space, therefore it constitutes a necessary and sufficient problem for stabilization problem. For any given plant, if there exists a solution to problem (3.14) which yields a zero objective, then one can claim that there exists a controller for the given plant.

Problem given in (3.14) is not in the form of classical $\mathcal{H}_2$ problem. One can benefit the vectorization method as shown in [37] to have the objective function in the form of $||\bar{H} + \bar{U}\bar{Q}||_2^2$ wherein $\bar{Q}$ is variable which can be solved using existing solution methods of $\mathcal{H}_2$ problem where a solution can be obtained in infinite dimensional space. However, vectorization method is computationally burdensome. Therefore, in the next section, we will provide an equivalent necessary and sufficient problem of (3.14) which can be solved to have a solution in infinite dimensional space without requiring vectorization method.

3.2.1 A Two-Step Procedure for Controller Synthesis Problem

In the previous section, we have defined the stabilization problem with variables $\tilde{Q}_1$ and Q_4. In this section by benefiting orthogonal spaces of AB_u and C_y, we will eliminate the variable Q_4. Afterwards, we will provide a two-step solution procedure for controller synthesis problem to obtain a controller in infinite dimensional space without a need of vectorization method.

Corollary 3. *Let the plant be as given in (2.11). Let A, B_u and C_y be such that null spaces of $(AB_u)^T$ and C_y are not empty. Let L_T be a concatenation of null space vectors of $(AB_u)^T$ and define $L := L_T^T$. Let R be a concatenation of null space vectors of C_y. There exists an internally stabilizing controller of P, if and only if there exists a $\tilde{Q}_1$ which makes the objective of the following problem zero.*

$$\begin{aligned} \min_{\tilde{Q}_1} \quad & \left\|-LA^2 + L(z^{-1}A - I)\tilde{Q}_1\right\|_2^2 + \left\|-A^2R + \tilde{Q}_1(z^{-1}A - I)R\right\|_2^2 \\ s.t. \quad & \tilde{Q}_1 \in \mathcal{RH}_\infty \end{aligned} \tag{3.19}$$

Moreover, let $\tilde{Q}_1^*$ be a solution to (3.19) such that objective of (3.19) is zero, and let $Q_4^* = (AB_u)^\dagger(-A^2(z^{-1}A - I) + (z^{-1}A - I)\tilde{Q}_1^*(z^{-1}A - I))C_y^\dagger$, then an internally stabilizing can be given as $K = -Q_4^*(I - P_{22}Q_4^*)^{-1}$.

Proof. A feasible solution to (3.17) must satisfy the followings

$$\begin{aligned} & -LA^2(z^{-1}A - I) + L(z^{-1}A - I)\tilde{Q}_1(z^{-1}A - I) = 0, \\ & -A^2(z^{-1}A - I)R + (z^{-1}A - I)\tilde{Q}_1(z^{-1}A - I)R = 0. \end{aligned} \tag{3.20}$$

Moreover, equality constraints in (3.20) can be simplified to

$$-LA^2 + L(z^{-1}A - I)\tilde{Q}_1 = 0, \tag{3.21a}$$

$$-A^2R + \tilde{Q}_1(z^{-1}A - I)R = 0. \tag{3.21b}$$

By regarding corollary 2, there exists an internally stabilizing controller if and only if there exists a feasible solution to (3.17) or equivalently to (3.21). Therefore, there exists an internally stabilizing controller if and only if there exists a solution to (3.19) such that its objective is zero and results follow corollary 2. □

Remark 1. *It should be noted that when null space of* $(AB_u)^T$ *or null space of* C_y *is empty, objective function of problem* (3.19) *reduces to* $\left\|-LA^2 + L(z^{-1}A - I)\tilde{Q}_1\right\|_2^2$ *or* $\left\|-A^2R + \tilde{Q}_1(z^{-1}A - I)R\right\|_2^2$. *Controller problems for these special cases can be found in appendix B.2.*

Problem in (3.19) allows one to solve the problem in smaller dimensions, in terms of both variable and constraint set.

As it can be noticed, problem given in (3.19) is not still in the form of $||H + UQV||_2^2$, hence classical $\mathcal{H}_2$ problem solution methods can not be applied. One may take advantage of the vectorization method as shown in [37] to put the problem (3.19) into form of $||\bar{H} + \bar{U}\bar{Q}||_2^2$ wherein $\bar{Q}$ is variable to benefit from the existing solution methods of $\mathcal{H}_2$ problem. Besides, we will show next that problem (3.19) can also be solved as two-step $\mathcal{H}_2$ problem to avoid the vectorization method since it is computationally burdensome.

Theorem 3. *Let* P *be as given in* (2.11). *Let* A, B_u *and* C_y *be such that null spaces of* $(AB_u)^T$ *and* C_y *are not empty. Let* L_T *be a concatenation of null space vectors of* B_u^T *and define* $L := L_T^T$. *Let* R *be a concatenation of null space vectors of* C_y. *Let* $\tilde{Q}_1^*$ *be a solution of the following problem if there exists which makes its objective zero.*

$$\begin{aligned} \min_{\tilde{Q}_1} \quad & \left\|-LA^2 + L(z^{-1}A - I)\tilde{Q}_1\right\|_2^2 \\ s.t. \quad & \tilde{Q}_1 \in \mathcal{RH}_\infty \end{aligned} \tag{3.22}$$

Let b *be column rank of* AB_u *and let* n_x *be order of* P_{22}. *Moreover, let* $W \in \mathcal{RH}_\infty^{n_x \times b}$ *be a any stable transfer function satisfying* $L(z^{-1}A - I)W = 0$ *such that* $W^\dagger W = I$. *Let* q^* *be a solution of the following problem if there exists any which makes its objective zero.*

$$\begin{aligned} \min_{q} \quad & \left\|-A^2R + \tilde{Q}_1^*(z^{-1}A - I)R + Wq(z^{-1}A - I)R\right\|_2^2 \\ s.t. \quad & q \in \mathcal{RH}_\infty^{b \times n_x} \end{aligned} \tag{3.23}$$

There exists an internally stabilizing controller, if and only if there exist such $\tilde{Q}_1^*$ and q^*. Moreover, for defined $\tilde{Q}_1 := \tilde{Q}_1^* + Wq^*$ and $Q_4 := (AB_u)^\dagger(-A^2(z^{-1}A - I) + (z^{-1}A - I)\tilde{Q}_1(z^{-1}A - I))C_y^\dagger$, one can obtain an internally stabilizing controller as $K = -Q_4(I - P_{22}Q_4)^{-1}$.

Proof. One need to have a feasible solution to (3.21) to obtain a controller according to corollary 3. To obtain $\tilde{Q}_1$, we can solve

$$-LA^2 - L(z^{-1}A - I)\tilde{Q}_1^* = 0 \tag{3.24}$$

Since this equation is an affine subspace, all the other $\tilde{Q}_1$s satisfying this equation must be such that

$$\tilde{Q}_1 = \tilde{Q}_1^* + \tilde{Q}_1^0 \tag{3.25}$$

where $L(z^{-1}A - I)\tilde{Q}_1^0 = 0$. This means $\tilde{Q}_1^0$ must be in the span of W where

$$L(z^{-1}A - I)W = 0. \tag{3.26}$$

Therefore, we can express $\tilde{Q}_1^0$ as $\tilde{Q}_1^0 = Wq$, so we can obtain $\tilde{Q}_1$ as $\tilde{Q}_1 = \tilde{Q}_1^* + Wq$ using equation (3.25). Among these $\tilde{Q}_1$s we want to find one such that it solves (3.21b), so, by substituting $\tilde{Q}_1 = \tilde{Q}_1^* + Wq$ into (3.21b) we obtain

$$-A^2R + (\tilde{Q}_1^* + Wq)(z^{-1}A - I)R = 0. \tag{3.27}$$

Therefore, equalities in (3.21) can be satisfied if and only if there exist feasible solutions to problems (3.22) and (3.23) such that their objectives hold zero norm, therefore results follow corollary 3. □

Remark 2. *Let A and B_u be such that null space of AB_u is not empty. Let b be column rank of AB_u and let n_x be order of P. Let $W_0 \in \mathbb{R}^{n_x \times b}$ be any matrix with full column rank and let β be a Lagrange multiplier high enough. A W satisfying $L(z^{-1}A - I)W = 0$ such that $W^\dagger W = I$ can be found by solving the following problem*

$$\min_W \quad \left\| \begin{bmatrix} \beta L(z^{-1}A - I)W \\ W - W_0 \end{bmatrix} \right\|_2^2. \tag{3.28}$$

Problem given in (3.28) can be equivalently written as in the following form which can be solved using existing solution methods of $\mathcal{H}_2$ problem to have a solution in infinite dimensional space.

$$\min_{W} \quad \left\| \begin{bmatrix} 0 \\ -W_0 \end{bmatrix} + \begin{bmatrix} \beta L(z^{-1}A - I) \\ I \end{bmatrix} W \right\|_2^2 . \tag{3.29}$$

It should be noted that minimization function $||W - W_0||_2^2$ functions as disturbance to have a W such that $W^{\dagger}W = I$. Moreover, β should be chosen high enough to have the constraint $L(z^{-1}A - I)W = 0$ satisfied.

Problems defined in (3.22) and (3.23) can be solved using existing solution methods of $\mathcal{H}_2$ optimal problem to have a solution in infinite dimensional space since their objectives holds the form of $\|H + UQV\|_2^2$ wherein Q is variable. Therefore, if there exists a controller, one can obtain a controller using the procedure described in theorem 3.

3.3 Special Cases

In this section we will provide some special cases of output feedback problem. We will first introduce stabilizability and detectability test problems. Afterwards, we will provide dynamic state-feedback and observer problems.

3.3.1 Stabilizability and Detectability Conditions

Herein, we introduce the necessary and sufficient conditions for stabilizability and detectability.

Lemma 3. *Plant in (2.11) or the pair (A, B_u) is stabilizable if and only if there exist casual $\bar{Q}_1$ and $\bar{Q}_3$ which make the objective of the following problem zero.*

$$\begin{aligned} \min_{\bar{Q}_1,\bar{Q}_3} \quad & \left\| I - \begin{bmatrix} z^{-1}A - I & z^{-1}B_u \end{bmatrix} \begin{bmatrix} \bar{Q}_1 \\ \bar{Q}_3 \end{bmatrix} \right\|_2^2 \\ s.t. \quad & \bar{Q}_1 \in \mathcal{RH}_\infty, \bar{Q}_3 \in \mathcal{RH}_\infty. \end{aligned} \tag{3.30}$$

Proof of lemma 3 can be found at appendix A.3. Lemma 3 provides a necessary and sufficient problem to test the stabilizability of the given plant. Next, we will provide a necessary and sufficient problem to test the detectability of the given plant.

Lemma 4. *Plant in (2.11) or pair (A, C_y) is detectable if and only if there exist $\bar{Q}_1$ and $\bar{Q}_2$ which make the objective of the following problem zero.*

$$\begin{aligned} \min_{\bar{Q}_1,\bar{Q}_2} \quad & \left\| I - \begin{bmatrix} \bar{Q}_1 & \bar{Q}_2 \end{bmatrix} \begin{bmatrix} z^{-1}A - I \\ z^{-1}C_y \end{bmatrix} \right\|_2^2 \\ s.t. \quad & \bar{Q}_1 \in \mathcal{RH}_\infty, \bar{Q}_2 \in \mathcal{RH}_\infty. \end{aligned} \tag{3.31}$$

Proof of lemma 4 can be found at appendix A.4.

In this section, we have formulated necessary and sufficient stabilizability and detectability conditions as $\mathcal{H}_2$ problems. Provided problems can be solved using existing $\mathcal{H}_2$ problem solution techniques to have a solution in infinite dimensional space.

3.3.2 Stabilizability and Detectability Conditions in Reduced Number of Variable

In this section, we will show that stabilizability and detectability problems given in corollary 3 and corollary 4 can be formulated with reduced number of variables benefiting orthogonal spaces of B_u and C_y.

Corollary 4. *Let plant P be as given in 2.11. Let B_u be such that null space of B_u^T is not empty. Let L_T be a concatenation of null space vectors of B_u^T and define $L := L_T^T$. Plant P or the pair (A, B_u) is stabilizable if and only if there exists a $\hat{Q}_1$ which makes the objective of the following problem zero.*

$$\begin{aligned} \min_{\hat{Q}_1} \quad & \left\| -LA + L(z^{-1}A - I)\hat{Q}_1 \right\|_2^2 \\ s.t. \quad & \hat{Q}_1 \in \mathcal{RH}_\infty. \end{aligned} \tag{3.32}$$

Proof of corollary 4 can be found in appendix A.5. It should be noted that for the case of null space of B_u^T is empty, given plant is trivially stabilizable.

Corollary 5. *Let plant P be as given in 2.11. Let C_y be such that null space of C_y is not empty. Let R be a concatenation of null space vectors of C_y. Plant P or the pair (A, C_y) is detectable if and only if there exists a $\hat{Q}_1$ which makes the objective of the following problem zero.*

$$\begin{aligned} \min_{\hat{Q}_1} \quad & \left\| -AR + \hat{Q}_1(z^{-1}A - I)R \right\|_2^2 \\ s.t. \quad & \hat{Q}_1 \in \mathcal{RH}_\infty. \end{aligned} \tag{3.33}$$

Proof of corollary 5 can be found in appendix A.6. It should be noted that for the case of null space of C_y is empty, given plant is trivially detectable.

Problems given in (3.32) and (3.33) are in the form of classical $\mathcal{H}_2$ problems and can be solved using existing $\mathcal{H}_2$ problem solution methods to have a solution in infinite dimensional space.

3.3.3 Dynamic State Feedback Controller and State Observer Problems

In this section, we will provide necessary and sufficient dynamic state-feedback controller and state observer problems benefiting all stabilizing controller parametrization defined for stably defined plants.

Let $\bar{P}_f$ be defined as

$$\bar{P}_f = \begin{bmatrix} z^{-1}A & z^{-1}B_u \end{bmatrix} \tag{3.34}$$

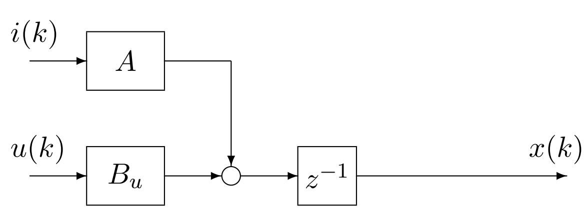

Figure 3.3 Block diagram of $\bar{P}_f$

A stable feedback interconnection of dynamic state-feedback interconnection of F and $P_{22} = \mathrm{ss}(A, B_u, C_y, 0)$ with $C_y = I$ equivalently means a stable feedback interconnection of $\bar{P}_f$ and $\bar{F} = \begin{bmatrix} I_{n_x} \\ F \end{bmatrix}$. Benefiting this stably defined plant and its controller, next we will show a necessary and sufficient problem to have a dynamic state-feedback controller.

Lemma 5. *Let plant, P, be defined as in (2.11). There exists a casual dynamic state-feedback controller of P, if and only if there exist casual $\bar{Q}_1$ and $\bar{Q}_3$ which make the objective of the following problem zero.*

$$\begin{aligned} \min_{\bar{Q}_1, \bar{Q}_3} \quad & \left\| I - \begin{bmatrix} z^{-1}A - I & z^{-1}B_u \end{bmatrix} \begin{bmatrix} \bar{Q}_1 \\ \bar{Q}_3 \end{bmatrix} \right\|_2^2 \\ s.t. \quad & \bar{Q}_1 \in \mathcal{RH}_\infty, \ \bar{Q}_3 \in \mathcal{RH}_\infty. \end{aligned} \tag{3.35}$$

Moreover, let $\bar{Q}_1$ and $\bar{Q}_3$ be a solution to (3.35) such that its objective is zero, then a state observer can be synthesized as $F = \bar{Q}_3 \bar{Q}_1^{-1}$.

Proof of lemma 5 can be found in appendix A.7.

Now, we will demonstrate the problem to formulate the state-observer problem. Let $\bar{P}_o$ be defined as

$$\bar{P}_o = \begin{bmatrix} z^{-1}A \\ z^{-1}C_y \end{bmatrix} \tag{3.36}$$

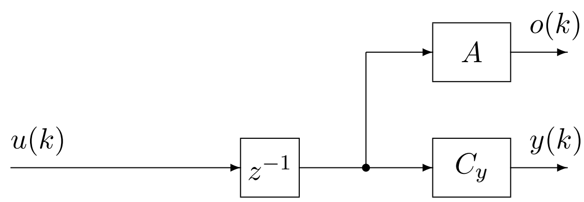

Figure 3.4 Block diagram of $\bar{P}_o$

A stable feedback interconnection of dynamic state-feedback interconnection of L and $P_{22} = \text{ss}(A, B_u, C_y, 0)$ with $B_u = I_{n_x}$ equivalently means a stable feedback interconnection of $\bar{P}_o$ and $\bar{L} = \begin{bmatrix} I_{n_x} & L \end{bmatrix}$. Benefiting this stably defined plant and its controller, next we will state a necessary and sufficient problem to have a dynamic state observer.

Lemma 6. *Let plant be defined as in* (2.11)*. There exists a casual dynamic state observer, if and only if there exist casual* $\bar{Q}_1$ *and* $\bar{Q}_2$ *which make the objective of the following problem zero.*

$$\begin{aligned} \min_{\bar{Q}_1,\bar{Q}_2} \quad & \left\| I - \begin{bmatrix} \bar{Q}_1 & \bar{Q}_2 \end{bmatrix} \begin{bmatrix} z^{-1}A - I \\ z^{-1}C_y \end{bmatrix} \right\|_2^2 \\ s.t. \quad & \bar{Q}_1 \in \mathcal{RH}_\infty, \ \bar{Q}_2 \in \mathcal{RH}_\infty. \end{aligned} \tag{3.37}$$

Moreover, let $\bar{Q}_1$ *and* $\bar{Q}_2$ *be a solution to* (3.37) *such that its objective is zero, then a state observer can be synthesized as* $L = \bar{Q}_1^{-1}\bar{Q}_2$*.*

Proof of lemma 6 can found in appendix A.8.

In this section, we provided necessary and sufficient dynamic state-feedback and state observer problems which are defined as classical $\mathcal{H}_2$ problems and can be solved using existing methods to have a solution in infinite dimensional space.

3.3.4 Dynamic State-Feedback and State Observer Problems in Reduced Number of Variable

In this section, we will provide state-feedback controller and state observer problems in reduced variables by regarding lemma 5 and 6 and orthogonal spaces of B_u and C_y.

Corollary 6. *Let plant* P *be as given in* 2.11*. Let* B_u *be such that null space of* B_u^T *is not empty. Let* L_T *be a concatenation of null space vectors of* B_u^T *and define* $L := L_T^T$*. There exists a casual dynamic state-feedback controller of* P*, if and only if there exists a casual* $\hat{Q}_1$ *which makes the objective of the following problem zero.*

$$\begin{aligned} \min_{\hat{Q}_1} \quad & \left\| -LA + L(z^{-1}A - I)\hat{Q}_1 \right\|_2^2 \\ s.t. \quad & \hat{Q}_1 \in \mathcal{RH}_\infty. \end{aligned} \tag{3.38}$$

Moreover, let $\bar{Q}_3 = B_u^\dagger(A - (z^{-1}A - I)\hat{Q}_1)$*, then state-feedback controller can be synthesized as* $F = \bar{Q}_3(z^{-1}\hat{Q}_1 - I)^{-1}$*.*

Proof of corollary 6 can be found in appendix A.9. Corollary 6 provides a necessary and sufficient problem to attain a dynamic state-feedback controller for the given plant.

Remark 3. *In the case of null space of B_u^T is empty, dynamic state feedback controllers of P can be trivially parameterized with a stable $\hat{Q}_1 \in \mathcal{RH}_\infty^{n_x \times n_x}$ as $F = \bar{Q}_3(z^{-1}\hat{Q}_1 - I)^{-1}$ wherein $\bar{Q}_3 = B_u^\dagger(A - (z^{-1}A - I)\hat{Q}_1)$.*

Corollary 7. *Let plant P be as given in 2.11. Let C_y be such that null space of C_y is not empty. Let R be a concatenation of null space vectors of C_y. There exists a casual dynamic state observer of P, if and only if there exists a casual $\hat{Q}_1$ which makes the objective of the following problem zero.*

$$\begin{aligned} \min_{\hat{Q}_1} \quad & \left\| -AR + \hat{Q}_1(z^{-1}A - I)R \right\|_2^2 \\ s.t. \quad & \hat{Q}_1 \in \mathcal{RH}_\infty. \end{aligned} \tag{3.39}$$

Moreover, let $\bar{Q}_2 = (A - \hat{Q}_1(z^{-1}A - I))C_y^\dagger$, then state observer can be synthesized as $L = (z^{-1}\hat{Q}_1 - I)^{-1}\bar{Q}_2$.

Proof of corollary 7 can be found in appendix A.10. Corollary 7 provides a necessary and sufficient problem to attain a dynamic state observer for the given plant.

Remark 4. *In the case of null space of C_y is empty, dynamic state observers of P can be trivially parameterized with stable $\hat{Q}_1 \in \mathcal{RH}_\infty^{n_x \times n_x}$ as $L = (z^{-1}\hat{Q}_1 - I)^{-1}\bar{Q}_2$ wherein $\bar{Q}_2 = (A - \hat{Q}_1(z^{-1}A - I))C_y^\dagger$.*

Problems defined in corollaries 6 and 7 include fewer variables with respect to state feedback controller and state observer problems defined in lemmas 5 and 6. Moreover, problems defined in (3.38) and (3.39) are defined in the form of $\mathcal{H}_2$ problem and can be solved using existing solution methods of $\mathcal{H}_2$ problem to have a solution in infinite dimensional space. Therefore, defined dynamic state-feedback controller and state observer problems constitute necessary and sufficiency problems to attain a dynamic state feedback controller and state observer.

3.4 Conclusion

In this chapter, we have obtained all stabilizing controller parametrization for any stable/unstable plant, by benefiting all stabilizing controller parametrization defined for stable plants, wherein constraints on Youla parameter are defined with equality constraints. Formulated all stabilizing controller parametrization does not necessitate to have a doubly coprime factorization or it does not require to have an initial controller.

Moreover, in this chapter we provided necessary and sufficient problems for the followings:

- Stabilizability test problem,
- Detectability test problem,
- Dynamic state-feedback controller problem,
- Dynamic state observer problem,
- Output feedback controller problem.

Stabilizability test problem, detectability test problem, dynamic state-feedback controller problem, dynamic output feedback problem are formulated in the form of $\mathcal{H}_2$ problem, i.e. in the form of $||H+UQV||_2^2$ for some H, U and V wherein Q is variable. Therefore, one can obtain a solution in infinite dimensional space by solving existing solution methods of $\mathcal{H}_2$ problem. Furthermore, we have provided a two-step solution procedure for output feedback controller problem to have a solution in infinite dimensional space.

Moreover, alternative controller problems can be found in appendix B.3 which are suffcient controller problems which do not require to use vectorization method and can be solved in one step using existing solution methods of $\mathcal{H}_2$ problem.

It should be noted that derivation of these controllers stands as base for the characterization of network distributed controller problems that will be introduced in next chapters.

CHAPTER 4. NETWORK IMPLEMENTABLE CONTROLLERS

By network implementable controllers, we refer to controllers with network implementable state-space realizations. Moreover, we call a controller as network realizable if there exists a network implementable state-space realization of it over the given network. In this chapter, we will define the set of network realizable controllers and will provide a method to obtain a network implementable state space realization of a given network realizable controller.

We first introduce a doubly coprime factorization of $\bar{K} := \textbf{blkdiag}(I_{n_x}, K)$ where K is a controller for the given plant P with order n_x. This doubly coprime factorization allows us to have left and right coprimes belong to stable network realizable set when K inherits the sparsity and delay constraints of the given network, i.e. $K \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$. We are able to obtain network implementable state-space realization of $K \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$, benefiting network implementable state-space realizations of coprimes of $\bar{K}$, since there exists a method to obtain network implementable state-space realization of a given stable network realizable system.

4.1 Doubly Coprime Factorization of Controllers

In this section, we will show that for a given controller, K, of a plant P with order n_x, $\textbf{blkdiag}(I_{n_x}, K)$ admits a doubly coprime factorization.

Since $\bar{P}_{22}$ given in (3.3) is stable, one can parametrize a controller $\bar{K}$ of $\bar{P}_{22}$ with a stable $\bar{Q}$ as $\bar{K} = -\bar{Q}(I - \bar{P}_{22}\bar{Q})^{-1}$. Thereupon, for a given controller K of P_{22}, one can obtain a stable $\bar{Q}$ as $\bar{Q}^* = -\bar{K}(I - \bar{P}_{22}\bar{K})^{-1}$ wherein $\bar{K} = \textbf{blkdiag}(I_{n_x}, K)$, which allows one to re-derive $\bar{K}$ as $\bar{K}^* = -\bar{Q}^*(I - \bar{P}_{22}\bar{Q}^*)^{-1}$ such that $\bar{K}^* = \mathbf{blkdiag}(I_{n_x}, K)$. This parametrization allows us to have right coprime factors of $\bar{K} = \mathbf{blkdiag}(I_{n_x}, K)$ which are $\bar{Q}$ and $I - \bar{P}_{22}\bar{Q}$. This knowledge allows us to state coprime factors of $\bar{K}$ as in the following lemma.

Lemma 7. *Let K be a stabilizing controller for $P_{22} = \boldsymbol{ss}(A, B_u, C_y, 0)$ and n_x be the order of P_{22}. Let $\bar{K}$ be defined as*

$$\bar{K} = \begin{bmatrix} I_{n_x} & 0 \\ 0 & K \end{bmatrix}. \tag{4.1}$$

Define $Z := (zI - A - B_u K C_y)$ and let set of maps V, W, $\bar{V}$ and $\bar{W}$ be defined as follows

$$\begin{aligned} V = \bar{V} &= \begin{bmatrix} I + AZ^{-1} & AZ^{-1}B_u K \\ KC_y Z^{-1} & K(I - P_{22}K)^{-1} \end{bmatrix}, \\ W &= \begin{bmatrix} I + AZ^{-1} & AZ^{-1}B_u K \\ C_y Z^{-1} & (I - P_{22}K)^{-1} \end{bmatrix}, \\ \bar{W} &= \begin{bmatrix} I + AZ^{-1} & AZ^{-1}B_u \\ KC_y Z^{-1} & (I - KP_{22})^{-1} \end{bmatrix}. \end{aligned} \tag{4.2}$$

Then, a doubly-coprime factorization of $\bar{K}$ can be represented as $\bar{K} = VW^{-1} = \bar{W}^{-1}\bar{V}$ satisfying

$$\Phi = \begin{bmatrix} \bar{X} & -\bar{Y} \\ -\bar{V} & \bar{W} \end{bmatrix} \begin{bmatrix} W & Y \\ V & X \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{4.3}$$

with stable X, Y, $\bar{X}$, $\bar{Y}$:

$$\begin{aligned} X &= \bar{X} = I, \\ Y = \bar{Y} &= \begin{bmatrix} z^{-1}A & z^{-1}AB_u \\ z^{-1}C_y & z^{-1}C_y B_u \end{bmatrix}. \end{aligned} \tag{4.4}$$

Proof of lemma 7 can be found in appendix A.11.

Provided coprime factors of $\bar{K} = \mathbf{blkdiag}(I_{n_x}, K)$ allow us to parametrize $\bar{K}$ with stable systems which will further allow us to obtain network implementable controllers of given network realizable controllers as it will be shown in next section.

4.2 Network Implementable Controllers

In this section, we will show that benefiting doubly-coprime factorization of controllers, we can obtain network implementable state-space realizations of controllers which inherits the sparsity and delay structures of the given graph, i.e. $K \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$.

The following corollary allows us to parametrize a given controller using two stable systems.

Corollary 8. *For a given plant $P_{22} = \boldsymbol{ss}(A, B_u, C_y, 0)$, let $\bar{P}_{22}$ and V be defined as in (3.3) and (4.2), respectively. Then, controller K of the given plant can be parametrized as*

$$K = \begin{bmatrix} 0 & I_{n_u} \end{bmatrix} V(I + \bar{P}_{22}V)^{-1} \begin{bmatrix} 0 \\ I_{n_y} \end{bmatrix}. \tag{4.5}$$

Proof. This corollary is direct result of lemma 7 and the equality $W = (I + \bar{P}_{22}V)$. □

In the following lemma, we show that there exists a network implementable state-space realization of a controller which inherits the sparsity and delay constraints of the given network, i.e. $K \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$, of the given plant $P_{22} = \mathbf{ss}(A, B_u, C_y, 0)$ wherein $A \in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_x, \mathcal{P}_x)$, $B_u \in S(I, \mathcal{P}_x, \mathcal{P}_u)$ and $C_y \in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_y, \mathcal{P}_x)$.

Lemma 8. *Let $A \in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_x, \mathcal{P}_x)$, $B_u \in S(I, \mathcal{P}_x, \mathcal{P}_u)$ and $C_y \in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_y, \mathcal{P}_x)$ be state-space matrices of strictly proper plant P_{22}, i.e. $P_{22} = \boldsymbol{ss}(A, B_u, C_y, 0)$, let n_x, n_u and n_y be number of states, inputs and outputs of the given plant, respectively. Let $K \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$ be a controller of P_{22}. Let $\tilde{V} \in \mathfrak{S}^s(\mathcal{G}^2, \mathcal{P}_{x_V}, [\mathcal{P}_x; \mathcal{P}_u], [\mathcal{P}_x; \mathcal{P}_y])$ be a network implementable state-space realization of V defined in (4.2). Define a network implementable $\tilde{\bar{P}}_{22}$ as*

$$\tilde{\bar{P}}_{22} = \boldsymbol{ss}(0, \begin{bmatrix} I_{n_x} & B_u \end{bmatrix}, \begin{bmatrix} A \\ C_y \end{bmatrix}, 0). \tag{4.6}$$

Then, a network implementable state-space realization of $K \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$, such that $\tilde{K} \in \mathfrak{S}(\mathcal{G}, \mathcal{P}_{x_K}, \mathcal{P}_u, \mathcal{P}_y)$, can be given as

$$\tilde{K} = \begin{bmatrix} 0 & I_{n_u} \end{bmatrix} \tilde{V}(I + \tilde{\bar{P}}_{22}\tilde{V})^{-1} \begin{bmatrix} 0 \\ I_{n_y} \end{bmatrix}. \tag{4.7}$$

Proof. For a given $K \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$, V given in (4.4) belongs to set $\mathfrak{T}^s(\mathcal{G}^2, [\mathcal{P}_x; \mathcal{P}_u], [\mathcal{P}_x; \mathcal{P}_y])$. According to theorem 1, there exists a network implementable state-space realization of $V \in \mathfrak{T}^s(\mathcal{G}^2, [\mathcal{P}_x; \mathcal{P}_u], [\mathcal{P}_x; \mathcal{P}_y])$ and let $\tilde{V}$ be a network implementable state-space realization of $V \in \mathfrak{T}^s(\mathcal{G}^2, [\mathcal{P}_x; \mathcal{P}_u], [\mathcal{P}_x; \mathcal{P}_y])$. Moreover, $\tilde{\bar{P}}_{22}$ given in (4.6) is a network implementable state-space realization of $\bar{P}_{22}$ given in (3.3a). Using corollary 8, we can parametrize the given controller as in (4.5). Therefore, using network implementable state-space realizations $\tilde{V}$ and $\tilde{\bar{P}}_{22}$, we can obtain a network-implementable controller, $\tilde{K}$, as in (4.7) equivalently, as in the block diagram exists in figure 4.1. □

Lemma 8 allows us to define the set of network implementable controllers and furthermore, it introduces a method to attain a network implementable state space realization for the given structured controllers $K \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$. A block diagram of network implementable realization of controller $K \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$ can be given as in figure 4.1, wherein $\tilde{V}$ is a network implementable state-space realization of V defined in (4.2) and $\tilde{\bar{P}}_{22}$ as in (4.6).

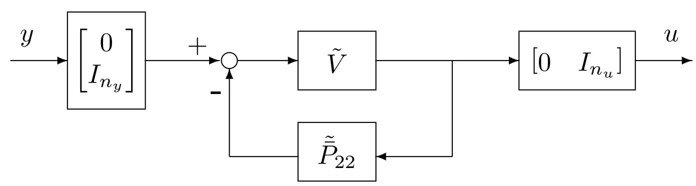

Figure 4.1 A block diagram for network realization of K.

Definition 9. *(Network realizable controller) We call a controller K of P_{22} as network realizable controller if there exists a network implementable state-space realization of it over the given network $\mathcal{G}$.*

One can obtain a network implementable state-space realization of $V \in \mathfrak{T}^s(\mathcal{G}^2, [\mathcal{P}_x, \mathcal{P}_u], [\mathcal{P}_x, \mathcal{P}_y])$ using methods in [1], [19] (one can refer to appendix C.1 for a review of method [1] and appendix C.2 for an example demonstration). Having network implementable state-space realizations $\tilde{V}$ and $\tilde{\bar{P}}_{22}$, one can attain a network implementable controller $\tilde{K} \in \mathfrak{S}(\mathcal{G}, \mathcal{P}_K, \mathcal{P}_u, \mathcal{P}_y)$ using the block diagram exists in figure 4.1. Therefore, a controller $K \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$ of a given networked system P_{22} as in (2.11) can be called as *network realizable controller* of P_{22}.

Theorem 4. *Let $A \in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_x, \mathcal{P}_x)$, $B_u \in S(I, \mathcal{P}_x, \mathcal{P}_u)$ and $C_y \in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_y, \mathcal{P}_x)$ be state-space matrices of strictly proper plant P_{22}, i.e. $P_{22} = \boldsymbol{ss}(A, B_u, C_y, 0)$. Let $K \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$ be a controller for P_{22}. Then, $K \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$ is a network realizable controller of P_{22}.*

Proof. Since, there exists a network implementable state-space realization of a controller $K \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$ of defined plant P_{22} according to lemma 8, it follows that K is a network realizable controller of P_{22}. □

Theorem 4 allows us to define the set of network realizable controllers. A necessary and sufficient problem to attain a network realizable controller for the given networked plant will be provided in the next chapter.

4.3 Conclusion

In this chapter, we have obtained a doubly coprime factorization of $\bar{K} = \mathbf{blkdiag}(I, K)$ wherein K is a controller of the given plant. Moreover, an alternative doubly coprime factorization of $\bar{K}$ can be found in appendix B.4. When K inherits the sparsity and delay constraints of the given network in z-domain, i.e. $K \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$, formulated coprime factors of $\bar{K}$ are stable network realizable systems. Hence, we are able to obtain network implementable state-space realization of coprime factors $\bar{K}$ benefiting existing network implementable state-space realization technique shown in [1] for stable network realizable systems (A review of network implementable state-space realization of stable network realizable systems has been provided in appendix C.1). By obtaining network implementable state-space realizations of coprime factors, we are able to obtain a network implementable state space realization of $\bar{K}$ and by proper mapping on $\bar{K}$, we are able to obtain a network implementable state-space realization of K. A demonstration of network implementable state-space realization of a network realizable controller can be found in appendix C.2.

Since, we are able to obtain network implementable state-space realizations of controllers which inherits the delay and sparsity constraints of the given network, we are able to define these controllers as network realizable controllers as given in theorem 4.

Obtained network implementable state-space realization method for network realizable controllers allows us to obtain network implementable controllers with reduced order with respect to existing realization methods. Comparisons with existing realization methods has been provided in numerical example sections 8.1 and 8.2.

CHAPTER 5. NETWORK REALIZABLE CONTROLLER PROBLEM

In this chapter, we obtain all stabilizing network realizable controller parametrization by regarding all stabilizing controller parametrization obtained in chapter 3. Moreover, we state a necessary and sufficient problem to have a network realizable controller which is a constrained problem. Afterwards, we provide its equivalent unconstrained network realizable controller problem which can solved using standard techniques to have a solution in infinite dimensional space.

5.1 All Stabilizing Network Realizable Controller Parametrization

In this section, we will provide all stabilizing network realizable controller parametrization benefiting lemma 2.

Lemma 9. *Let the networked plant, P, be given as in (2.11), such that $P_{22} \in \mathfrak{S}(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_y, \mathcal{P}_u)$ and let $\bar{Q} := \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix}$. All stabilizing network realizable controllers of P, K, such that $K(z) \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$ can be parametrized as $K = -Q_4(I - P_{22}Q_4)^{-1}$ where $\bar{Q}$ satisfies equality constraints in (3.4) such that $Q_4(z) \in \mathfrak{T}^s(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$ and $\bar{Q} \in \mathcal{RH}_\infty$.*

Proof. It is proven in lemma 2 that for a $\bar{Q} \in \mathcal{RH}_\infty$ satisfying (3.4), $K = -Q_4(I - P_{22}Q_4)^{-1}$ parametrizes all stabilizing controllers. Next, we will show that $K(z)$ belongs to set $\mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$ if and only if we have $Q_4(z) \in \mathfrak{T}^s(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$.

$\Rightarrow$ First assume Q_4 is stable and network realizable, i.e. $Q_4(z) \in \mathfrak{T}^s(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$, then we have $K = -Q_4(I - P_{22}Q_4)^{-1}$ such that $K(z) \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$ which is a network realizable controller according to theorem 4.

$\Leftarrow$ Now assume, we have $K(z) \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$. Let $\bar{K}^* = \mathbf{blkdiag}(I_{n_x}, K)$. Since we have $P_{22} \in \mathfrak{S}(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_y, \mathcal{P}_u)$, $\bar{P}_{22}$ defined in (3.3a) belongs to set $\mathfrak{T}^s(\bar{\mathcal{G}}, [\mathcal{P}_x; \mathcal{P}_y], [\mathcal{P}_x; \mathcal{P}_u])$. Therefore, we have $\bar{Q}^* = -\bar{K}^*(I - \bar{P}_{22}\bar{K}^*)^{-1}$ such that $\bar{Q}^*(z) \in \mathfrak{T}^s(\bar{\mathcal{G}}, [\mathcal{P}_x, \mathcal{P}_u], [\mathcal{P}_x, \mathcal{P}_y])$ when $K(z) \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$. Therefore, we have $Q_4^*(z) = \begin{bmatrix} 0 & I_{n_u} \end{bmatrix} \bar{Q}^*(z) \begin{bmatrix} 0 \\ I_{n_y} \end{bmatrix} \in \mathfrak{T}^s(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$. □

Lemma 9 shows that one can parametrize all stabilizing network realizable controllers with a stable $\bar{Q}$ satisfying equality constraints 3.4 with an additional structural constraint on Q_4 as $Q_4 \in \mathfrak{T}^s(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$.

5.2 Network Realizable Controller Problem

In this section, we will first state a necessary and sufficient network realizable controller problem. Afterwards, we will derive its equivalent unconstrained problem which can be solved with existing solution methods of $\mathcal{H}_2$ problem to have a solution in infinite dimensional space.

Theorem 5. *Let plant, P, be given as in (2.11) such that $P_{22} \in \mathfrak{S}(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_y, \mathcal{P}_u)$. There exists a network realizable internally stabilizing controller K such that $K(z) \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$, if and only if there exist $\tilde{Q}_1$ and Q_4 which make the objective of following problem zero.*

$$\begin{aligned} \min_{\tilde{Q}_1, Q_4} & \left\| A^2(z^{-1}A - I) - (z^{-1}A - I)\tilde{Q}_1(z^{-1}A - I) + AB_uQ_4C_y \right\|_2^2 \\ s.t. \quad & \tilde{Q}_1 \in \mathcal{RH}_\infty, \; Q_4(z) \in \mathfrak{T}^s(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y) \end{aligned} \tag{5.1}$$

Moreover, let $\tilde{Q}_1^*$ and Q_4^* be solution of (5.1) such that its objective is zero, then a network realizable controller can be constructed as $K = -Q_4^*(I - P_{22}Q_4^*)^{-1}$ such that $K(z) \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$.

Proof. As it is proved in the proof of corollary 2, equality constraints in (3.4) can be equivalently solved with the equality constraint in (3.16). Therefore, regarding equivalent constraint sets (3.4) and (3.16), and lemma 9, there exists a network realizable controller if and only if there exist Q_1^* and Q_4^* which make objective of problem 5.1 zero and corresponding network realizable controller can be given as $K = -Q_4^*(I - P_{22}Q_4^*)^{-1}$. □

Problem defined in (5.1) is a necessary and sufficient problem, however it is a constrained problem, therefore, classical $\mathcal{H}_2$ solution methods cannot be applied. Next, we will benefit from vectorization method to define an equivalent unconstrained problem which is in the form of $\mathcal{H}_2$ problem wherein a solution can be obtained in infinite dimensional space using existing solution methods.

Let $\mathbf{vec}(\tilde{Q}_1(z)) = \bar{W}_1(z)$ and $\mathbf{vec}(Q_4(z)) = S_4(z)\bar{W}_4(z)$ be the vectorized elements of $\bar{Q}_1$ and Q_4 where $\bar{W}_1(z) \in \mathcal{RH}_\infty^{n_x^2 \times 1}$, $\bar{W}_4(z) \in \mathcal{RH}_\infty^{a \times 1}$ for some a and $S_4(z) \in \mathcal{RH}_\infty^{n_u n_y \times a}$ contains the delay and sparsity constraints imposed by the set $\mathfrak{T}^s(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$ (for a demonstration of how to obtain systems $S_4(z)$ and $\bar{W}_4(z)$ one can refer to section B.5). Benefiting results of vectorization, we write an equivalent of problem (5.1) as in the following

$$\min_{\bar{W}_1, \bar{W}_4} \left\| A^2(z^{-1}A - I) - \begin{bmatrix} (z^{-1}A - I)^T \otimes (z^{-1}A - I) & -(C_y^T \otimes AB_u)S_4 \end{bmatrix} \begin{bmatrix} \bar{W}_1 \\ \bar{W}_4 \end{bmatrix} \right\|_2^2 \quad (5.2)$$

$$\text{s.t.} \quad \bar{W}_1 \in \mathcal{RH}_\infty, \bar{W}_4 \in \mathcal{RH}_\infty.$$

Let $\bar{W}_1^*$ and $\bar{W}_4^*$ be solution of problem (5.2), such that its objective is zero, then one can obtain Q_4^* by $Q_4^* = \mathbf{vec}(S_4\bar{W}_4^*)^{-1}$ and optimal controller as $K^* = -Q_4^*(I - P_{22}Q_4^*)^{-1}$ wherein $K(z) \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$. After obtaining a network realizable controller, we obtain a network implementable controller using the network implementable state-space realization technique formulated in section 4.2, i.e. using block diagram in figure 4.1 where $\tilde{V}$ is network implementable state-space realization of V which is defined in (4.2) (V is function of K^*.) and $\tilde{\bar{P}}_{22}$ is as defined in (4.6).

Problem (5.2) is an unconstrained necessary and sufficient network realizable controller problem and it is in the form of $\mathcal{H}_2$ problem. Therefore, problem (5.2) can be solved using existing solution techniques of $\mathcal{H}_2$ problem to attain a solution in infinite-dimensional space. Hence, if there exists a network realizable controller, one can obtain it by solving (5.2).

Moreover, using the network realizable controller obtained in this section, one can solve the optimal network realizable controller problem in theorem 9 to obtain a optimal network realizable controller. Once the optimal network realizable controller problem in theorem 9 is solved, then one can form an optimal network realizable controller as in (7.9). After obtaining an optimal network realizable controller, one can obtain a network implementable controller using the network implementable state-space realization technique formulated in section 4.2.

5.3 Conclusion

In this chapter, first, we obtained all stabilizing network realizable controller parametrization, then we provided necessary and sufficient network realizable controller problem.

By benefiting vectorization method, we defined a network realizable control problem in the form of unconstrained $\mathcal{H}_2$ problem wherein a solution can be obtained in infinite dimensional space benefiting existing solution methods of $\mathcal{H}_2$ problem. After obtaining a network realizable controller, we obtain a network implementable controller using the network implementable state-space realization method formulated in section 4.2. Moreover, a demonstration of how to obtain a network implementable state-space realization of network realizable controller can be found in appendix C.2.

CHAPTER 6. OPTIMAL NETWORK REALIZABLE CONTROLLER PROBLEM

In literature, optimal controller problem have been studied with sufficiency conditions. In [37], optimal network realizable controller problem has been formulated as a function of static sparse state feedback and static state observer. However, methods to find static sparse state feedback and static state observer have been provided with sufficient problems. Moreover, in the work of [43], optimal network distributed controller problem has been defined with infinite dimensional constraints with not know a priory finite support solution. In this section, we will define infinite dimensional optimal network distributed controller problem which can be solved benefiting existing solution methods of $\mathcal{H}_2$ problem.

We will first provide a model matching problem, then we will formulate optimal network realizable controller problem. Afterwards, we will formulate an unconstrained optimal network realizable controller problem which can be solved benefiting existing solution methods to have solution in infinite dimensional space. The provided optimal network realizable controller problem allows one to solve and obtain a solution in infinite dimensional space and does not necessitate priory computations such as doubly-coprime factorization of plant or an initial controller unlike optimal controller problems involve well-known Youla parametrization.

6.1 Model Matching Problem

In this section, we formulate a model matching problem affine in $\bar{Q}$ to be able to define a convex optimal controller problem.

We had formulated $\bar{P}_{22}$ in the previous chapters. Now, generalized congruent plant $\bar{P}$ can be given as

$$\begin{aligned}\bar{P} &= \left[\begin{array}{c|c} \bar{P}_{11} & \bar{P}_{12} \\ \hline \bar{P}_{21} & \bar{P}_{22} \end{array}\right] \\ &= \left[\begin{array}{c|cc} z^{-1}C_zB_w + D_{zw} & z^{-1}C_z & z^{-1}C_zB_u + D_{zu} \\ \hline z^{-1}AB_w & z^{-1}A & z^{-1}AB_u \\ z^{-1}C_yB_w + D_{yw} & z^{-1}C_y & z^{-1}C_yB_u \end{array}\right], \end{aligned} \tag{6.1}$$

then feedback interconnection of $\bar{P}$ and $\bar{K}$ is equivalent to feedback interconnection of P and K. Moreover, a block diagram of $\bar{P}$ can be found in figure 6.1.

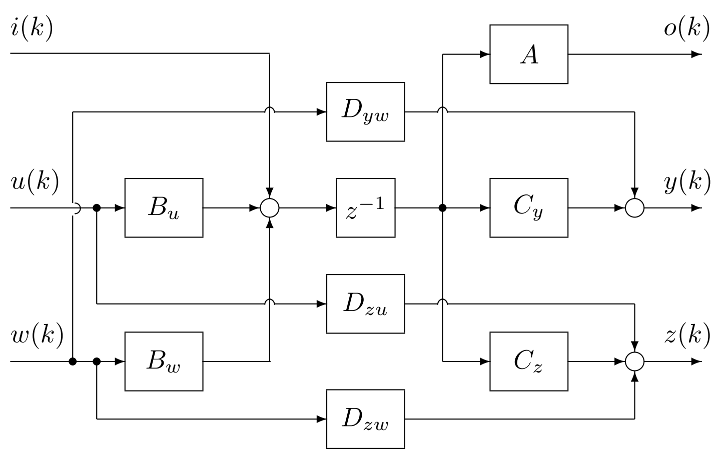

Figure 6.1 Block diagram of $\bar{P}$.

Input to output system from w to z can be given as

$$T_{zw} = \bar{P}_{11} + \bar{P}_{12}\bar{K}(I - \bar{P}_{22}\bar{K})^{-1}\bar{P}_{21}. \tag{6.2}$$

Since, for a controller parametrization $\bar{K} = -\bar{Q}(I - \bar{P}_{22}\bar{Q})^{-1}$ we have $\bar{Q} = -\bar{K}(I - \bar{P}_{22}\bar{K})^{-1}$, closed loop parametrization in (6.2) can be given as

$$T_{zw} = \bar{P}_{11} - \bar{P}_{12}\bar{Q}\bar{P}_{21}. \tag{6.3}$$

Input to output map given in (6.3) is affine in $\bar{Q}$ which will allow us to define the optimal controller problem convex in Youla parameter.

6.2 Optimal Network Realizable Controller Problem

In this section, we define a necessary and sufficient problem to obtain optimal network realizable controllers benefiting the network realizable controller problem defined in previous chapter.

Theorem 6. *Let plant, P, be given as in (2.11) such that $P_{22} \in \mathfrak{S}(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_y, \mathcal{P}_u)$. Let $\bar{P}_{11}$, $\bar{P}_{12}$ and $\bar{P}_{21}$ be defined as in (6.1). Let $\bar{Q} := \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix}$. Let $\bar{Q}^*$ be a solution of the following problem,*

$$\begin{aligned} \min_{\bar{Q}} \quad & ||\bar{P}_{11} - \bar{P}_{12}\bar{Q}\bar{P}_{21}|| \\ s.t. \quad & \left\| \begin{bmatrix} I & 0 \end{bmatrix} - \begin{bmatrix} z^{-1}A - I & z^{-1}AB_u \end{bmatrix} \bar{Q} \right\| + \left\| \begin{bmatrix} I \\ 0 \end{bmatrix} - \bar{Q} \begin{bmatrix} z^{-1}A - I \\ z^{-1}C_y \end{bmatrix} \right\| = 0, \\ & \bar{Q} \in \mathcal{RH}_\infty, \; Q_4(z) \in \mathfrak{T}^s(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y) \end{aligned} \tag{6.4}$$

then optimal network realizable controller K such that $K(z) \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$ can be given as $K^* = -Q_4^*(I - P_{22}Q_4^*)^{-1}$.

Proof. Proof follows lemma 9. □

Problem (6.4) is necessary and sufficient optimal controller problem. However, it can not be solved in infinite dimensional space with the existing solution methods. In the next section, we define an unconstrained optimal controller problem which can be solved in infinite dimensional space.

6.2.1 Unconstrained Optimal Network Realizable Controller Problem

As it can be noticed from problem (6.4), it cannot be solved in infinite dimensional space using existing solution methods to obtain optimal controller due to equality constraint and, structural and sparsity constraints introduced on Q_4 by the space $\mathfrak{T}^s(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$. Next, we introduce a Lagrange variable and benefit from the vectorization method to be able to define an unconstrained optimal network realizable controller problem.

Benefiting corollary 1, problem given in (6.4) can be formulated as in the following with a Lagrange variable β, wherein β is big enough, so that equality constraints in problem 6.4 is satisfied.

$$\begin{aligned} \min_{\bar{Q}_1} \quad & ||\bar{P}_{11} - \bar{P}_{12}\bar{Q}\bar{P}_{21}|| \\ & + \beta \left\| \begin{bmatrix} I & 0 \end{bmatrix} - \begin{bmatrix} z^{-1}A - I & z^{-1}AB_u \end{bmatrix} \bar{Q} \right\| + \beta \left\| Q_3(z^{-1}A - I) + z^{-1}Q_4 C_y \right\| \\ \text{s.t.} \quad & \bar{Q} \in \mathcal{RH}_\infty, \; Q_4(z) \in \mathfrak{T}^s(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y). \end{aligned} \tag{6.5}$$

We can append sum of norms into rows of one minimization function as follows

$$\begin{aligned} \min_{\bar{Q}} \quad & \left\| \begin{matrix} \bar{P}_{11} - \bar{P}_{12}\bar{Q}\bar{P}_{21} \\ \beta(I - (z^{-1}A - I)Q_1 - z^{-1}AB_uQ_3) \\ \beta(-(z^{-1}A - I)Q_2 - z^{-1}AB_uQ_4) \\ \beta(-Q_3(z^{-1}A - I) - z^{-1}Q_4C_y) \end{matrix} \right\| \\ \text{s.t.} \quad & \bar{Q} \in \mathcal{RH}_\infty. \end{aligned} \tag{6.6}$$

Let $\bar{U}_{11} = z^{-2}(AB_w)^T \otimes C_z$, $\bar{U}_{12} = z^{-1}(z^{-1}C_yB_w + D_{yw})^T \otimes C_z$, $\bar{U}_{13} = z^{-1}(AB_w)^T \otimes (z^{-1}C_zB_u + D_{zu})$, $\bar{U}_{14} = (z^{-1}C_yB_w + D_{yw})^T \otimes (z^{-1}C_zB_u + D_{zu})$ and $Z_A = z^{-1}A - I$. Let $\mathbf{vec}(Q_4(z)) = S_4(z)\bar{W}_4(z)$ be the vectorized element of Q_4 where $\bar{W}_4(z) \in \mathcal{RH}_\infty^{a\times 1}$ and $S_4(z)$ contains the delay and sparsity constraints imposed by the set $\mathfrak{T}(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$ (For a demonstration of how to obtain systems $S_4(z)$ and $\bar{W}_4(z)$ one can refer to section B.5.). Let β be a Lagrange variable big enough. Benefiting a Lagrange multiplier and vectorization method, we re-define the problem (6.4) as follows

$$\begin{aligned} \min_{\bar{W}} \quad & \left\| \begin{bmatrix} \mathbf{vec}(z^{-1}C_zB_w + D_{zw}) \\ \beta\mathbf{vec}(I) \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \bar{U}_{11} & \bar{U}_{12} & \bar{U}_{13} & \bar{U}_{14} \\ \beta I \otimes Z_A & 0 & z^{-1}\beta I \otimes (AB_u) & 0 \\ 0 & \beta I \otimes Z_A & 0 & z^{-1}\beta I \otimes (AB_u) \\ 0 & 0 & \beta Z_A^T \otimes I & z^{-1}\beta C_y^T \otimes I \end{bmatrix} \begin{bmatrix} \bar{W}_1 \\ \bar{W}_2 \\ \bar{W}_3 \\ \bar{W}_4 \end{bmatrix} \right\| \\ \text{s.t.} \quad & \bar{Q} \in \mathcal{RH}_\infty. \end{aligned} \tag{6.7}$$

Problem (6.7) can be solved in infinite dimensional space using existing solution methods of $\mathcal{H}_2$ problem to have an $\mathcal{H}_2$ optimal network realizable controller. For existing solution techniques of $\mathcal{H}_2$ problem, one may refer to [4], [5]. Moreover, provided optimal controller problem can be used to obtain ℓ_1 optimal network realizable controller using ℓ_1 problem solution technique introduced in [17].

Let $\bar{W}_4^*$ be a solution of problem (6.7). By applying de-vectorization process, we obtain Q_4 as $Q_4^* := \mathbf{vec}^{-1}(S_4\bar{W}_4^*)$, let the solution be such that equality constraint in problem (6.4) is satisfied, then an optimal network realizable controller can be found by $K^* = -Q_4^*(I - P_{22}Q_4^*)^{-1}$ wherein $K(z) \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$. After obtaining a network realizable controller, we obtain a network implementable controller using the network implementable state-space realization technique formulated in section 4.2, i.e. using block diagram in figure 4.1 where $\tilde{V}$ is network implementable state-space realization of V which is defined in (4.2) (V is function of K^*.) and $\tilde{\bar{P}}_{22}$ is as defined in (4.6).

Remark 5. *If a solution to (6.7) is not satisfying the equality constraint in problem (6.4), one may increase Lagrange multiplier β to satisfy the constraint with a negligible error.*

6.2.2 Optimal Network Realizable Control Problem in Reduced Variables

In this section, we define a necessary and sufficient optimal network realizable controller in reduced variables.

Benefiting theorem 5, we define an optimal network realizable problem in reduced variable as in the following corollary.

Corollary 9. *Let plant, P, be given as in (2.11) such that $P_{22} \in \mathfrak{S}(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_y, \mathcal{P}_u)$. Let A, B_u and C_y be such that AB_u has full column rank and C_y has full row rank. Let $\bar{P}_{11}$, $\bar{P}_{12}$ and $\bar{P}_{21}$ be defined as in (6.1), $\bar{P}_{11}^{Q_1}$, $\bar{P}_{12}^{Q_1}$, and $\bar{P}_{21}^{Q_1}$ be defined as in the following*

$$\bar{P}_{11}^{Q_1} := \begin{bmatrix} -I - z^{-1}A & z^{-1}A^2C_y^{\dagger} \\ z^{-1}(AB_u)^{\dagger}A^2 & -(AB_u)^{\dagger}A^2(z^{-1}A - I)C_y^{\dagger} \end{bmatrix},$$
$$\bar{P}_{12}^{Q_1} := \begin{bmatrix} z^{-1}I \\ -(AB_u)^{\dagger}(z^{-1}A - I) \end{bmatrix}, \quad \bar{P}_{21}^{Q_1} := \begin{bmatrix} z^{-1}I & -(z^{-1}A - I)C_y^{\dagger} \end{bmatrix}. \tag{6.8}$$

Let $\tilde{Q}_1^*$ and Q_4^* be a solution of the following problem,

$$\begin{aligned} \min_{\tilde{Q}_1, Q_4} \quad & ||\bar{P}_{11} - \bar{P}_{12}\bar{P}_{11}^{Q_1}\bar{P}_{21} - \bar{P}_{12}\bar{P}_{12}^{Q_1}\tilde{Q}_1\bar{P}_{21}^{Q_1}\bar{P}_{21}|| \\ s.t. \quad & \left\| A^2(z^{-1}A - I) - (z^{-1}A - I)\tilde{Q}_1(z^{-1}A - I) + AB_uQ_4C_y \right\| = 0, \\ & \tilde{Q}_1 \in \mathcal{RH}_\infty, \; Q_4(z) \in \mathfrak{T}^s(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y). \end{aligned} \tag{6.9}$$

then optimal network realizable controller K such that $K(z) \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$ can be given as $K^* = -Q_4^*(I - P_{22}Q_4^*)^{-1}$.

Proof. Using equation (3.12a) and $Q_1 = -I - z^{-1}A + z^{-2}\tilde{Q}_1$, we obtain Q_2 in terms of $\tilde{Q}_1$ as follows

$$Q_2 = z^{-1}(A^2 - \tilde{Q}_1(z^{-1}A - I))C_y^{\dagger}. \tag{6.10}$$

Furthermore, using (3.7a) and $Q_1 = -I - z^{-1}A + z^{-2}\tilde{Q}_1$, we obtain Q_3 in terms of $\tilde{Q}_1$ as follows

$$Q_3 = z^{-1}(AB_u)^{\dagger}(A^2 - (z^{-1}A - I)\tilde{Q}_1). \tag{6.11}$$

Using (3.17), Q_4 can be defined in terms of $\tilde{Q}_1$ as follows

$$Q_4 = (AB_u)^{\dagger}(-A^2(z^{-1}A - I) + (z^{-1}A - I)\tilde{Q}_1(z^{-1}A - I))C_y^{\dagger}. \tag{6.12}$$

Using (6.10), (6.11) and (6.12), we can define $\bar{Q}$ in terms of $\tilde{Q}_1$ as follows

$$\begin{aligned} \bar{Q} &= \begin{bmatrix} -I - z^{-1}A + z^{-2}A\tilde{Q}_1 & z^{-1}(A^2 - \tilde{Q}_1P_c)C_y^{\dagger} \\ z^{-1}(AB_u)^{\dagger}(A^2 - P_c\tilde{Q}_1) & B_u^{\dagger}(-A^2P_c + P_c\tilde{Q}_1P_c)C_y^{\dagger} \end{bmatrix} \\ &= \bar{P}_{11}^{Q_1} + \bar{P}_{12}^{Q_1}\tilde{Q}_1\bar{P}_{21}^{Q_1} \end{aligned} \tag{6.13}$$

Regarding theorem 5, we can put the constraint to have a network realizable controller as zero norm of its objective with constraints $\tilde{Q}_1 \in \mathcal{RH}_\infty$ and $Q_4 \in \mathfrak{T}^s(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$, wherein for such Q_4, controller can be synthesized as $K = -Q_4(I - P_{22}Q_4)^{-1}$ such that $K(z) \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$ and results follow. $\square$

Remark 6. *It should be noted that when AB_u does not have column rank deficiency or C_y does not have row rank deficiency, problem (6.9) allows one to obtain optimal network realizable controller. Otherwise, due to pseudo inverse usage, one may obtain sub-optimal results. Therefore, one may benefit from problem defined in (6.4) to obtain an optimal network realizable controller when there is column rank deficiency in AB_u or row rank deficiency in C_y.*

Now, benefiting a Lagrange multiplier we can write optimal network realizable controller problem as follows

$$\begin{aligned} \min_{\tilde{Q}_1, Q_4} \quad & ||\bar{P}_{11} - \bar{P}_{12}\bar{P}_{11}^{Q_1}\bar{P}_{21} - \bar{P}_{12}\bar{P}_{12}^{Q_1}\tilde{Q}_1\bar{P}_{21}^{Q_1}\bar{P}_{21}|| + \\ & \beta \left\| A^2(z^{-1}A - I) - (z^{-1}A - I)\tilde{Q}_1(z^{-1}A - I) + AB_uQ_4C_y \right\| \\ \text{s.t.} \quad & \tilde{Q}_1 \in \mathcal{RH}_\infty, \; Q_4(z) \in \mathfrak{T}^s(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y). \end{aligned} \tag{6.14}$$

Lagrange multiplier β should be chosen high enough to satisfy the constraint

$$\left\| A^2(z^{-1}A - I) - (z^{-1}A - I)\tilde{Q}_1(z^{-1}A - I) + AB_uQ_4C_y \right\| = 0. \tag{6.15}$$

Moreover, problem (6.14) still can not be solved using existing solution methods to have a solution in infinite dimensional space due to structural constraint on Q_4. Therefore, we benefit vectorization method to define an unconstrained optimal network realizable controller problem.

Let $\mathbf{vec}(\tilde{Q}_1(z)) = \bar{W}_1(z)$ and $\mathbf{vec}(Q_4(z)) = S_4(z)\bar{W}_4(z)$ be the vectorized elements of $\tilde{Q}_1$ and Q_4 where $\bar{W}_1(z) \in \mathcal{RH}_\infty^{n_x^2 \times 1}$, $\bar{W}_4(z) \in \mathcal{RH}_\infty^{a \times 1}$ and $S_4(z)$ contains the delay and sparsity constraints imposed by the set $\mathfrak{T}(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$ (For a demonstration of how to obtain systems $S_4(z)$ and $\bar{W}_4(z)$ one can refer to section B.5.). Let $Z_A = z^{-1}A - I$. Now, we can define the vectorized form of problem (6.14) as in the following problem which is an unconstrained problem and can be solved in infinite-dimensional space.

$$\min_{\bar{W}_1,\bar{W}_4} \left\| \begin{bmatrix} \mathbf{vec}(\bar{P}_{11} - \bar{P}_{12}\bar{P}_{11}^{Q_1}\bar{P}_{21}) \\ \beta A^2(z^{-1}A - I) \end{bmatrix} - \begin{bmatrix} (\bar{P}_{21}^{Q_1}\bar{P}_{21})^T \otimes (\bar{P}_{12}\bar{P}_{12}^{Q_1}) & 0 \\ \beta Z_A^T \otimes Z_A & -\beta(C_y^T \otimes AB_u)S_4 \end{bmatrix} \begin{bmatrix} \bar{W}_1 \\ \bar{W}_4 \end{bmatrix} \right\|$$
$$s.t. \quad \bar{W}_1 \in \mathcal{RH}_\infty, \quad \bar{W}_4 \in \mathcal{RH}_\infty \tag{6.16}$$

Let $\bar{W}_4^* \in \mathcal{RH}_\infty$ be the solution to problem (6.16) such that

$$\left\| A^2(z^{-1}A - I) - \begin{bmatrix} Z_A^T \otimes Z_A & -(C_y^T \otimes AB_u)S_4 \end{bmatrix} \begin{bmatrix} \bar{W}_1 \\ \bar{W}_4 \end{bmatrix} \right\| = 0 \tag{6.17}$$

is satisfied, let $Q_4^* = \mathbf{vec}(S_4\bar{W}_4^*)^{-1}$ and then one can obtain an optimal realizable controller by $K^* = -Q_4^*(I - P_{22}Q_4^*)^{-1}$ such that $K(z) \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$. After obtaining a network realizable controller, we obtain a network implementable controller using the network implementable state-space realization technique formulated in section 4.2, i.e. using block diagram in figure 4.1 where $\tilde{V}$ is network implementable state-space realization of V which is defined in (4.2) (V is function of K^*.) and $\tilde{\tilde{P}}_{22}$ is as defined in (4.6).

It should be noted that in order to obtain the internally-stabilizing controller, (6.17) need to be satisfied. Since (6.17) is a hard constraint for problem (6.16), β need to be chosen big enough.

Problem (6.16) can be solved in infinite dimensional space using existing solution methods of $\mathcal{H}_2$ problem to have an $\mathcal{H}_2$ optimal network realizable controller. For existing solution techniques of $\mathcal{H}_2$ problem, one may refer to [4], [5]. Moreover, provided optimal controller problem can be used to obtain ℓ_1 optimal network realizable controller using ℓ_1 problem solution technique introduced in [17].

6.3 Conclusion

In this chapter, we have defined a necessary and sufficient optimal network realizable controller problem which is a constrained optimization problem. Benefiting a Lagrange multiplier and vectorization method, we defined optimal network realizable controller problem in the form of unconstrained $\mathcal{H}_2$ optimal problem wherein a solution can be obtained in infinite dimensional space benefiting existing solution techniques of $\mathcal{H}_2$ problem. For existing solution techniques of $\mathcal{H}_2$ problem, one may refer to [4], [5]. Moreover, provided optimal controller problem can be used to obtain ℓ_1 optimal network realizable controller using ℓ_1 problem solution technique introduced in [17].

After obtaining an optimal network realizable controller, one can obtain a network implementable state-space realization benefiting the network implementable state-space realization technique formulated in section 4.2. A demonstration of how to obtain a network implementable controller can be found in appendix C.2. We provided a five-node and a six-node numerical examples in sections 8.1 and 8.2 wherein optimal network realizable controller problem formulated in this chapter has been solved and network implementable controller obtained using the network implementable state-space realization method in section 4.2. Moreover, we provided comparisons with other existing methods on those provided numerical example sections. Regarding the results of optimal network implementable controller orders obtained in sections 8.1 and 8.2, one can observe the efficiency of the optimal network realizable controller problem obtained in this chapter.

CHAPTER 7. SPECIAL RESULTS FOR STRONGLY CONNECTED SYSTEMS

The result of this section was obtained before those of the previous sections which are more general. Here, we decided to add them at this stage as they apply to the special case where a network realizable controller is given or can be easily found. This is the case for example of strongly connected networks.

In this chapter, we provide an alternative all stabilizing network realizable controller parametrization. We first review the existing all stabilizing controller parametrization which is a function of an initial controller. Afterwards, we provide all stabilizing network realizable controller parametrization as a function of any network realizable controller K_0 in the form of $-\mathbf{lft}(K_0, \mathbf{lft}(J, Q))$ wherein J is a stable system as a function of K_0 and Q is the Youla parameter. We propose network realizable controllers in the form of delayed controllers for strongly connected networks. This allows us to parametrize all stabilizing network realizable controllers of strongly connected networks. Besides, one can benefit the network realizable controller problem formulated in chapter 5 to have an initial network realizable controller to parametrize all stabilizing network realizable controllers.

Moreover, we obtain a model matching problem and define an optimal network realizable controller problem benefiting aforementioned all stabilizing controller parametrization. After solving optimal network realizable controller problem, one can obtain a network implementable state-space realization of it in two ways: 1) Obtain network implementable realization of K_0, J and Q and synthesize the optimal network implementable controller as $-\mathbf{lft}(\tilde{K}_0, \mathbf{lft}(\tilde{J}, \tilde{Q}))$ wherein $\tilde{K}_0$, $\tilde{J}$ and $\tilde{Q}$ are network implementable state space realization of K_0, J and Q ($\tilde{K}_0$ can be obtained as in section 4.2.). 2) First, obtain the optimal network realizable controller as $-\mathbf{lft}(K_0, \mathbf{lft}(J, Q))$, then obtain a network implementable state-space realization of this optimal controller benefiting the network implementable state-space realization technique formulated in section 4.2.

7.1 An Overview on Optimal Network Realizable Controllers

The approach we have proposed in the previous chapters provides a unified general and efficient way to obtain optimal network realizable controllers directly with network realizable variables. Most of previous efforts like [28], [37] require an initial structured controller to be found to solve optimal network realizable controller problem. In [28], an initial network realizable controller was needed. Extension of this approach would need: 1) a necessary and sufficient problem to obtain an initial network realizable controller, K_0, 2) a network implementable realization of K_0, 3) a convenient parametrization of all network realizable K's like $K = \mathbf{lft}(J, Q)$ where J is network realizable as a function of K_0. As it will be shown in the following sections, we are able to obtain all stabilizing controller parametrization as $-\mathbf{lft}(K_0, \mathbf{lft}(J, Q))$. So that, once the optimal network realizable controller problem is solved for a network realizable Q, optimal network implementable controller can be obtained by obtaining network implementable realizations of K_0, J and Q. Moreover, using the result of chapter 4, we can omit the step 2 (a network implementable realization of K_0) and we can avoid to derive a realization of the initial stabilizing controller, since we can obtain a network implementable controller after synthesizing the optimal network realizable controller.

Obtaining an initial network realizable controller can be found using the network realizable model based controller (MBC) as proposed by [29] wherein a network implementable realization technique of this MBC is also provided or using the network realizable controller problem proposed in chapter 5. There are also cases like the strongly connected networks where obtaining such controller is reasonably straightforward as shown later.

7.2 A Review on All Stabilizing Controllers as a Function of an Initial Controller

Herein, we review the all stabilizing controllers defined as a function of output feedback controllers following [27], [28].

Let K_0 be a stabilizing controller for the plant given in (2.11). It is shown in [40] that parallel plants, P and K_0, admit a doubly coprime factorization. Let the maps N_1, D_1, $\bar{N}_1$ and $\bar{D}_1$ be defined as follows

$$\begin{aligned} N_1 &= \begin{bmatrix} K_0(I - P_{22}K_0)^{-1} & K_0P_{22}(I - K_0P_{22})^{-1} \\ -P_{22}K_0(I - P_{22}K_0)^{-1} & -P_{22}(I - K_0P_{22})^{-1} \end{bmatrix}, \\ D_1 &= \begin{bmatrix} (I - P_{22}K_0)^{-1} & P_{22}(I - K_0P_{22})^{-1} \\ K_0(I - P_{22}K_0)^{-1} & (I - K_0P_{22})^{-1} \end{bmatrix}, \\ \bar{D}_1 &= \begin{bmatrix} (I - K_0P_{22})^{-1} & -K_0(I - P_{22}K_0)^{-1} \\ -P_{22}(I - K_0P_{22})^{-1} & (I - P_{22}K_0)^{-1} \end{bmatrix}, \\ \bar{N}_1 &= N_1. \end{aligned} \tag{7.1}$$

Then, a doubly coprime factorization of $P_1 = \mathbf{blkdiag}(K_0, -P_{22})$ can be given as $P_1 = N_1D_1^{-1} = \bar{D}_1^{-1}\bar{N}_1$ satisfying

$$\begin{bmatrix} \bar{X}_1 & -\bar{Y}_1 \\ -\bar{N}_1 & \bar{D}_1 \end{bmatrix} \begin{bmatrix} D_1 & Y_1 \\ N_1 & X_1 \end{bmatrix} = I, \tag{7.2}$$

with stable X_1, Y_1, $\bar{X}_1$, $\bar{Y}_1$:

$$\begin{aligned} X_1 &= \bar{X}_1 = I_{n_y+n_u}, \\ Y_1 &= \bar{Y}_1 = \begin{bmatrix} 0 & -I_{n_y} \\ I_{n_u} & 0 \end{bmatrix}. \end{aligned} \tag{7.3}$$

Theorem 7. *[27] Let P be stabilizable and detectable plant. If there exists a causal K_0 such that* $\textbf{\textit{lft}}(P, K_0)$ *is stable, then set of internally stabilizing all controllers for* $P_1 = \textbf{\textit{blkdiag}}(K_0, -P)$ *is parameterized by*

$$
\begin{aligned}
K_1(Q) &= (\bar{X}_1 - Q\bar{N}_1)^{-1}(\bar{Y}_1 - Q\bar{D}_1), \\
&= (Y_1 - D_1Q)(X_1 - N_1Q)^{-1}.
\end{aligned}
\tag{7.4}
$$

where X_1, Y_1, N_1, D_1, $\bar{X}_1$, $\bar{Y}_1$, $\bar{N}_1$, $\bar{D}_1$ are as in (7.1) and Q is stable. Moreover, all stabilizing controllers of P can be parametrized as $\mathbf{lft}(-K_0, -K_1(Q))$.

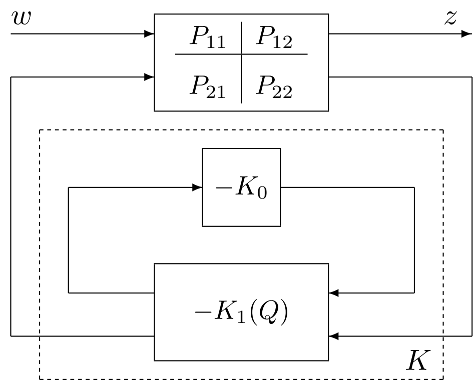

Figure 7.1 Block diagram of $K = \mathbf{lft}(-K_0, -K_1(Q))$ in feedback with plant P.

7.3 All Stabilizing Controller Re-parametrization

In this section, we re-parametrize the controller parametrization given in (7.4) to avoid the inverse operation in it.

Lemma 10. *Let N_1, D_1, $\bar{N}_1$, $\bar{D}_1$, X_1, Y_1, $\bar{X}_1$ and $\bar{Y}_1$ be the elements of Bezout-identity as in (7.2), then $K_1 = (Y_1 - D_1Q)(X_1 - N_1Q)^{-1}$ can be given as $K_1 = \mathbf{lft}(\bar{J}_1, Q)$ with a $\bar{J}_1$ defined as follows*

$$
\bar{J}_1 = \begin{bmatrix} \bar{Y}_1 & -I \\ I & N_1 \end{bmatrix}. \tag{7.5}
$$

Proof. Using the Bezout identity, one can obtain $D_1 = \bar{X}_1 + \bar{X}^{-1}\bar{Y}_1N_1$. Moreover, Y_1 can be expressed as $Y_1 = \bar{X}_1^{-1}\bar{Y}_1X_1$. Using these, K_1 can be re-parametrized as follows [5]

$$\begin{aligned} K_1 &= (Y_1 - D_1 Q)(X_1 - N_1 Q)^{-1} \\ &= (\bar{X}_1^{-1}\bar{Y}_1 X_1 - (\bar{X}_1 + \bar{X}^{-1}\bar{Y}_1 N_1)Q)(X_1 - N_1 Q)^{-1} \\ &= \bar{X}_1^{-1}(\bar{Y}_1(X_1 - N_1 Q) - Q)(X_1 - N_1 Q)^{-1} \\ &= \bar{X}_1^{-1}(\bar{Y}_1 - Q(X_1 - N_1 Q)^{-1}) \\ &= \bar{X}_1^{-1}\bar{Y}_1 - \bar{X}_1^{-1} Q(I - X_1^{-1} N_1 Q)^{-1} X_1^{-1}. \end{aligned} \tag{7.6}$$

$\bar{X}_1^{-1}\bar{Y}_1 - \bar{X}_1^{-1} Q(I - X_1^{-1} N_1 Q)^{-1} X_1^{-1}$ is nothing but $\mathbf{lft}(\bar{J}_1, Q)$ where $\bar{J}_1$ is defined as

$$\bar{J}_1 = \begin{bmatrix} \bar{X}_1^{-1}\bar{Y}_1 & -\bar{X}_1^{-1} \\ X_1^{-1} & X_1^{-1} N_1 \end{bmatrix}. \tag{7.7}$$

Using definition of X_1, $\bar{X}_1$ as in (7.3), we can simplify $\bar{J}_1$ as in (7.5). □

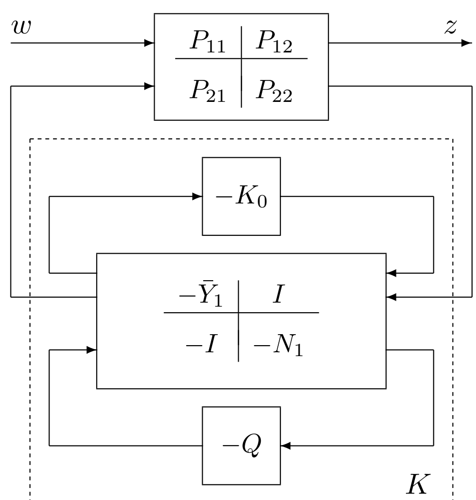

Figure 7.2 Block diagram of $K = \mathbf{lft}(-K_0, \mathbf{lft}(-\bar{J}_1, -Q))$ in feedback with plant P.

After solving model matching problem for a stable Q, one can construct a controller using $\bar{J}_1$ defined as in (7.5) as follows

$$K = \mathbf{lft}(-K_0, \mathbf{lft}(-\bar{J}_1, -Q)). \tag{7.8}$$

Block diagram of K defined in (7.8) can be seen in figure 7.2. All stabilizing controller parametrization given in (7.8) allows one to avoid the inverse operation given all stabilizing controller parametrization in (7.4).

7.4 All Stabilizing Network Realizable Controller Parametrization as a Function of Network Realizable Controllers

Herein, all stabilizing network realizable controllers are proposed as a function of network realizable controllers benefiting the all stabilizing controller parametrization given in (7.8).

Lemma 11. *Let the networked plant, P, as given in (2.11) which has a strictly causal interaction over the given graph $\mathcal{G}$. For a given network realizable controller $K_0 \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$ of P, the set of all stabilizing network realizable controllers for P can be parametrized with $Q \in \mathfrak{T}^s(\mathcal{G}^2, [\mathcal{P}_y; \mathcal{P}_u], [\mathcal{P}_u; \mathcal{P}_y])$ as follow*

$$K = \boldsymbol{lft}(-K_0, \boldsymbol{lft}(-\bar{J}_1, -Q)) \tag{7.9}$$

with $\bar{J}_1$ which is network realizable system defined as in (7.5) where $\bar{X}_1$, $\bar{Y}_1$, X_1 and N_1 are defined as in (7.1) as a function of $K_0 \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$.

Proof. K given in (7.9) parametrizes all stabilizing controllers for a given K_0 which stabilizes P_{22} with $Q \in \mathcal{RH}_\infty$ according to theorem 7 and in the view of lemma 10. Next, we show that Q belongs to set $\mathfrak{T}^s(\mathcal{G}^2, [\mathcal{P}_y; \mathcal{P}_u], [\mathcal{P}_u; \mathcal{P}_y])$ for a controller belongs to set $\mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$.

("$\supset$"): For a given plant $P_{22} \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$ and $K_0 \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$ which stabilizes P_{22}, we have $\bar{J}_1 \in \mathfrak{T}^s(\mathcal{G}^4, [\mathcal{P}_y; \mathcal{P}_u; \mathcal{P}_u; \mathcal{P}_y], [\mathcal{P}_u; \mathcal{P}_y; \mathcal{P}_y; \mathcal{P}_u])$ and when Q is in space $\mathfrak{T}^s(\mathcal{G}^2, [\mathcal{P}_y; \mathcal{P}_u], [\mathcal{P}_u; \mathcal{P}_y])$, we obtain a network realizable controller with controller parametrization $K = \mathbf{lft}(-K_0, \mathbf{lft}(-\bar{J}_1, -Q)) \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$.

("$\subset$"):

Since we have the parametrization $K = \mathbf{lft}(-K_0, \mathbf{lft}(-\bar{J}_1, -Q))$, we can equivalently parametrize K as $K = \mathbf{lft}(\mathbf{lft}(-K_0, -\bar{J}_1), -Q)$, let $J = \mathbf{lft}(-K_0, -\bar{J}_1)$, then we have $J \in \mathfrak{T}(\mathcal{G}^3, [\mathcal{P}_u; \mathcal{P}_u; \mathcal{P}_y], [\mathcal{P}_y; \mathcal{P}_y; \mathcal{P}_u])$, so controller can be parametrized as $K = \mathbf{lft}(J, -Q)$ where

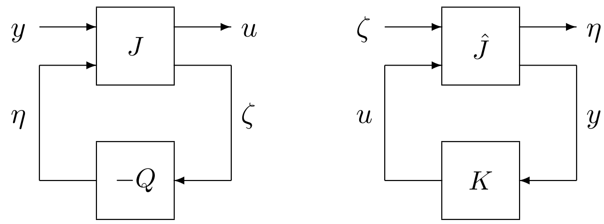

Figure 7.3 Block diagram of $K = \mathbf{lft}(J, -Q)$ and $Q = -\mathbf{lft}(\hat{J}, K)$.

Q is stable. Let $Q = \mathbf{lft}(-\hat{J}, -K)$. Using input-output relations given in figure 7.3, we obtain the following relationship between $J(z)$ and $\hat{J}(z)$

$$\begin{bmatrix} 0 & I_{n_u} & 0 \\ 0 & 0 & I_{n_y} \\ I_{n_u} & 0 & 0 \end{bmatrix} J(z) \begin{bmatrix} 0 & 0 & I_{n_y} \\ I_{n_y} & 0 & 0 \\ 0 & I_{n_u} & 0 \end{bmatrix} \hat{J}(z) = I. \tag{7.10}$$

Since we have $J(z)$ in set $\mathfrak{T}(\mathcal{G}^3, [\mathcal{P}_u; \mathcal{P}_u; \mathcal{P}_y], [\mathcal{P}_y; \mathcal{P}_y; \mathcal{P}_u])$, $\hat{J}(z)$ belongs to set $\mathfrak{T}(\mathcal{G}^3, [\mathcal{P}_y; \mathcal{P}_u; \mathcal{P}_y], [\mathcal{P}_u; \mathcal{P}_y; \mathcal{P}_u])$. Therefore, $Q = \mathbf{lft}(-\hat{J}, -K)$ belongs to set $\mathfrak{T}(\mathcal{G}^2, [\mathcal{P}_y; \mathcal{P}_u], [\mathcal{P}_u; \mathcal{P}_y])$ for a controller, K, belongs to set $\mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$. Since, $Q \in \mathcal{RH}_\infty$, it belongs to set $\mathfrak{T}^s(\mathcal{G}^2, [\mathcal{P}_y; \mathcal{P}_u], [\mathcal{P}_u; \mathcal{P}_y])$. □

Next section provides an optimal network realizable controller problem benefiting the network realizable controller parametrization given in lemma 11.

7.5 Network Realizable Controllers for Strongly Connected Networks

In this section, we show a necessary and sufficient problem to have a network realizable controller for strongly connected networks.

As it is shown in chapter 4, if a controller inherits the sparsity and delay constraints of a given graph in z-domain, then it has a network-implementable state-space realization. Since the strongly connected networked systems need to have only delay constraints in z-domain, one can have a network realizable controller in the form of delayed controllers. The following theorem sets the necessary and sufficient conditions to have a network realizable controller for strongly connected systems in the delayed system form.

Lemma 12. *Let plant P be defined as in (2.11) be stabilizable and detectable, where the graph $\mathcal{G}$ is strongly connected and let d be the longest path on $\mathcal{G}$. Let $K_i = ss(A_{K_i}, B_{K_i}, C_{K_i}, 0)$ be a strictly proper stabilizing controller for $z^{-(d-1)}P_{22}$. Then,*

$$K = z^{-(d-1)}K_i \tag{7.11}$$

is a network realizable controller for P_{22} on $\mathcal{G}$.

Proof. Feedback interconnection of P_{22} and K, i.e. $\mathbf{lft}(P_{22}, K)$, is stable if $(I - P_{22}K)^{-1}$ is stable. Considering the equality $(I - P_{22}(z^{-(d-1)}K_i))^{-1} = (I - (z^{-(d-1)}P_{22})K_i)^{-1}$, P_{22} is output feedback controllable if and only if $P_d = z^{-(d-1)}P_{22}$ is stabilizable and detectable, since the unstable poles of P_d belongs to the set of unstable poles of P_{22}. It is trivial to show the rank of observability matrix of P_d is $\text{rank}(\mathcal{O}_{P_d}) = (d-1)n + \text{rank}(\mathcal{O}_{P_{22}})$ and the rank of controllability matrix of P_d is $\text{rank}(\mathcal{C}_{P_d}) \geq \text{rank}(\mathcal{C}_{P_{22}})$, which completes the proof. □

Steps to find a delayed controller can be given as follows:

1. Let $(A_d, B_d, C_d, 0)$ be state-space matrices of $P_d = z^{-(d-1)}P_{22}$.
2. Find F such that $A_d + B_dF$ stable.
3. Find L such that $A_d + LC_d$ stable.
4. Construct K_i as
$$K_i : \left[\begin{array}{c|c} A_d + B_dF + LC_d & -L \\ \hline F & 0 \end{array}\right]. \tag{7.12}$$
5. Add delays to K_i: $K = z^{-(d-1)}K_i$.

Since, a delayed controller as given in (7.11) inherits the delay constraints in z-domain for a strongly connected network with longest path d, one can obtain a network implementable state-space realization of this delayed controller using the realization method introduced in section 4.2.

Theorem 8. *Let plant P be defined as in (2.11) be stabilizable and detectable, where the graph $\mathcal{G}$ is strongly connected, let d be the longest path on $\mathcal{G}$ and, let n_x, n_u and n_y be number of states, inputs and outputs of the given plant, respectively. Let $K_i = ss(A_{K_i}, B_{K_i}, C_{K_i}, 0)$ be a strictly proper stabilizing controller for $z^{-(d-1)}P_{22}$ and define $K := z^{-(d-1)}K_i$. Let $\tilde{V}_d \in \mathfrak{S}^s(\mathcal{G}^2, \mathcal{P}_{x_V}, [\mathcal{P}_x; \mathcal{P}_u], [\mathcal{P}_x; \mathcal{P}_y])$ be a network implementable state-space realization of V_d which is defined as*

$$V_d = \begin{bmatrix} I + A(zI - A - B_u K C_y)^{-1} & A(zI - A - B_u K C_y)^{-1} B_u K \\ K C_y (zI - A - B_u K C_y)^{-1} & K(I - P_{22}K)^{-1} \end{bmatrix},$$

and let $\tilde{\tilde{P}}_{22} = \boldsymbol{ss}(0, \begin{bmatrix} I_{n_x} & B_u \end{bmatrix}, \begin{bmatrix} A \\ C_y \end{bmatrix}, 0)$*, then* $\tilde{K} = \begin{bmatrix} 0 & I_{n_u} \end{bmatrix} \tilde{V}_d (I + \tilde{\tilde{P}}_{22} \tilde{V}_d)^{-1} \begin{bmatrix} 0 \\ I_{n_y} \end{bmatrix} \in \mathfrak{S}(\mathcal{G}, \mathcal{P}_{x_K}, \mathcal{P}_u, \mathcal{P}_y)$ *is a network implementable controller for the given strongly connected plant.*

Proof. Theorem 8 is a direct result of lemma 8 and lemma 12. □

Theorem 8 allows one to obtain network implementable controller for the given strongly connected networked plant.

7.6 Optimal Network Realizable Controller Problem as a Function of Network Realizable Controllers

In this section, we define a model matching problem and define an optimal network realizable control problem for network distributed systems as a function of an initial network realizable controller.

The $\mathcal{H}_2$ optimal controller problem to obtain a centralized controller can be described as follows

$$\begin{aligned} \min \quad & ||\mathbf{lft}(P,K)|| \\ \text{s.t.} \quad & K \text{ stabilizing.} \end{aligned} \tag{7.13}$$

Furthermore, we are interested in optimal network realizable controller. A network realizable controller should be an element in $\mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$. Therefore, we can define a optimal network realizable controller problem as

$$\begin{aligned} \min \quad & ||\mathbf{lft}(P,K)|| \\ \text{s.t.} \quad & K \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y), \quad K \text{ stabilizing.} \end{aligned} \tag{7.14}$$

In lemma 11, we parametrized the internally stabilizing network realizable controllers as a function of a network realizable Youla parameter. We are interested in solving optimal network realizable controller problem in a convex way. Next, we formulate a model matching problem to have the input to output map affine in Youla parameter.

7.6.1 Model Matching Problem

Herein, we provide a model matching problem to obtain input to output map affine in Q where Q parametrizes all stabilizing controllers as given in (7.8).

One can define a model matching problem with the Bezout identity elements. Let N, D, $\tilde{N}$ and $\tilde{D}$ be set of maps satisfy (2.15) such that $G_{22} = ND^{-1} = \tilde{D}^{-1}\tilde{N}$. Moreover, closed loop map from w to z can be given as

$$T_{zw} = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21} \tag{7.15}$$

using $K(I - G_{22}K)^{-1} = (Y - DQ)\tilde{D}$, closed loop map from w to z given in (7.15) can be written as follows [5]

$$
\begin{aligned}
T_{zw} &= H + UQV \\
H &= G_{11} + G_{12} Y \tilde{D} G_{21}, \\
U &= -G_{12} D, \\
V &= \tilde{D} G_{21}.
\end{aligned}
\tag{7.16}
$$

We benefit the doubly-coprime factorization of $P_1 = \mathbf{blkdiag}(K_0, -P_{22})$ where doubly coprime factors, N_1, D_1, $\tilde{N}_1$ and $\tilde{D}_1$ are defined as in (7.1) to define the model matching problem by defining generalized plant as

$$
G = \left[\begin{array}{c|cc} -P_{11} & 0 & -P_{12} \\ \hline 0 & K_0 & 0 \\ -P_{21} & 0 & -P_{22} \end{array}\right]
\tag{7.17}
$$

Now, we can define a model matching problem as follows using (7.16) and (7.17)

$$
\begin{aligned}
T_{zw} &= \bar{H} + \bar{U} Q \bar{V} \\
\bar{H} &= -P_{11} + \begin{bmatrix} 0 & -P_{12} \end{bmatrix} Y_1 \tilde{D}_1 \begin{bmatrix} 0 \\ -P_{21} \end{bmatrix}, \\
\bar{U} &= \begin{bmatrix} 0 & P_{12} \end{bmatrix} D_1, \\
\bar{V} &= \tilde{D}_1 \begin{bmatrix} 0 \\ -P_{21} \end{bmatrix}.
\end{aligned}
\tag{7.18}
$$

For a given stabilizing controller K_0 of plant P, using definitions of D_1 and $\bar{D}_1$ exist in (7.1), model matching problem can be defined as in the following with a stable Q.

$$
\begin{aligned}
T_{zw} &= T_{11} + T_{12} Q T_{21} \\
T_{11} &= -P_{11} - P_{12} K_0 (I - P_{22} K_0)^{-1} P_{21}, \\
T_{12} &= \begin{bmatrix} P_{12} K_0 (I - P_{22} K_0)^{-1} & P_{12} (I - K_0 P_{22})^{-1} \end{bmatrix}, \\
T_{21} &= \begin{bmatrix} K_0 (I - P_{22} K_0)^{-1} P_{21} \\ -(I - P_{22} K_0)^{-1} P_{21} \end{bmatrix}.
\end{aligned}
\tag{7.19}
$$

Input to output map defined in (7.19) allows us to have T_{zw} affine in Q which will allow us to define the optimal controller problem convex in Q.

7.6.2 Optimal Network Realizable Controller Problem

As shown in section 7.5, we can find a network realizable controller for strongly connected networks as given in (7.11). Moreover, one can obtain a network realizable controller for any network by solving necessary and sufficient network realizable controller problem given in (6.16) . Using a network realizable controller, we can define an optimal network realizable controller problem as in the following theorem benefiting the model matching problem formulated in (7.19).

Theorem 9. *Let the stabilizable and detectable networked plant, P be defined as in* (2.11) *and let $K_0 \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$ be a network realizable controller of P. Let T_{11}, T_{12} and T_{22} be given as in* (7.19) *as a function of K_0. Then, an optimal network realizable controller problem can be given as follows*

$$\begin{aligned} \min \quad & ||T_{11} + T_{12} Q T_{21}|| \\ s.t. \quad & Q \in \mathfrak{T}^s(\mathcal{G}^2, [\mathcal{P}_y; \mathcal{P}_u], [\mathcal{P}_u; \mathcal{P}_y]) \end{aligned} \tag{7.20}$$

let $Q^* \in \mathfrak{T}^s(\mathcal{G}^2, [\mathcal{P}_y; \mathcal{P}_u], [\mathcal{P}_u; \mathcal{P}_y])$ be a solution of (7.20)*, then an optimal network realizable controller can be synthesized as $K = \textbf{\textit{lft}}(-K_0, \textbf{\textit{lft}}(-\bar{J}_1, -Q^*))$ where and $\bar{J}_1$ is as defined in* (7.5) *as a function of K_0.*

Proof. Proof follows theorem 4 and lemma 11. □

Problem defined in (7.20) is non-convex due to structural constraints imposed on Q. Therefore, we benefit from the vectorization method as shown in [37]. Let $\mathbf{vec}(Q(z))$ be the vectorized elements of $Q(z) \in \mathfrak{T}^s(\mathcal{G}^2, [\mathcal{P}_y; \mathcal{P}_u], [\mathcal{P}_u; \mathcal{P}_y])$. One can write $\mathbf{vec}(Q(z))$ as $\mathbf{vec}(Q(z)) = S(z)W(z)$ for some $W(z) \in \mathcal{RH}_\infty^{a\times 1}$, where $S(z)$ contains the delay and sparsity constraints imposed by the set $\mathfrak{T}^s(\mathcal{G}^2, [\mathcal{P}_y; \mathcal{P}_u], [\mathcal{P}_u; \mathcal{P}_y])$ (For a demonstration of how to obtain systems $S(z)$ and $W(z)$, one can refer to section B.5.). Using the vectorization method as shown in [37], we obtain

$$||T_{11}(z) + T_{12}(z)Q(z)T_{21}(z)|| = ||\mathbf{vec}(T_{11}(z)) + (T_{21}^T(z) \otimes T_{12}(z))S(z)W(z)|| \tag{7.21}$$

Now, problem in (7.20) can be equivalently written as

$$\begin{aligned} \min \quad & ||\mathbf{vec}(T_{11}) + (T_{21}^T \otimes T_{12})SW|| \\ \text{s.t.} \quad & W \in \mathcal{RH}_\infty^{a\times 1} \end{aligned} \tag{7.22}$$

which is a convex problem and can be solved using standard techniques. Problem (7.22) allows one to obtain $\mathcal{H}_2$ optimal network realizable controller benefiting existing solution methods of $\mathcal{H}_2$ problem wherein the solution lies in infinite dimensional space. For existing solution techniques of $\mathcal{H}_2$ problem, one may refer to [4], [5]. Moreover, provided optimal controller problem can be used to obtain ℓ_1 optimal network realizable controller using ℓ_1 problem solution technique introduced in [17].

The procedure to obtain an optimal network implementable controller can be listed as in the following method.

Method 1:

1. Find $K_0 \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$ (If the given network is strongly connected network, one can find K_0 as in theorem 12. Otherwise, one may refer to network realizable controller problem in (6.16).).

2. Obtain T_{11}, T_{12} and T_{21} as given in (7.19).

3. Solve the problem in (7.22) for $W^* \in \mathcal{RH}_\infty$ and obtain $Q^* = \mathbf{vec}^{-1}(SW^*) \in \mathfrak{T}^s(\mathcal{G}^2, [\mathcal{P}_y; \mathcal{P}_u], [\mathcal{P}_u; \mathcal{P}_y])$.

4. Obtain a network implementable realization of K_0, $\tilde{K}_0$, using the block diagram in figure 4.1 where $\tilde{V}$ is network implementable state-space realization of V which is defined in (4.2) (V is function of K_0.) and $\tilde{\tilde{P}}_{22}$ is as defined in (4.6).

5. Obtain the network implementable state-space realizations $\tilde{Q}^*$ and $\tilde{\bar{J}}_1$ of Q^* and $\bar{J}_1$ where $\bar{J}_1$ is as in (7.5).

6. Obtain the optimal network implementable controller as $\tilde{K}^* = \mathbf{lft}(-\tilde{K}_0, \mathbf{lft}(-\tilde{\bar{J}}_1, -\tilde{Q}^*))$.

Remark 7. *Network implementable state-space realizations of stable network realizable systems can be obtained by the methods provided in [1] which is reviewed in section C.1.*

One can also obtain an optimal network implementable controller $\tilde{K}^*$ as described in the following method.

Method 2:

1. Apply the steps 1–3 in Method 1.

2. Obtain network realizable controller $K^* = \mathbf{lft}(-K_0, \mathbf{lft}(-\bar{J}_1, -Q^*)) \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$.

3. Obtain a network implementable state-space realization of network realizable K^* using network implementable state-space realization technique shown in section 4.2, i.e. using block diagram in figure 4.1 where $\tilde{V}$ is network implementable state-space realization of V which is defined in (4.2) (V is function of K^*.) and $\tilde{\bar{P}}_{22}$ is as defined in (4.6).

It should be noted that results obtained in numerical example sections indicates that one can obtain fewer controller order using Method 2 with respect to using Method 1.

7.7 A Comparison with Existing Optimal Network Realizable Controller Problems

Optimal network implementable controller obtained using the optimal network realizable controller problem described in theorem 9 has been compared with the controllers obtained with existing optimal network realizable controller problems in section 8.2. By regarding the results of optimal network implementable controller orders obtained in section 8.2, one can observe the efficiency of the optimal network realizable controller problem obtained in chapter 6. Optimal network realizable controller problem defined in theorem 9 is causing an order inflation due to starting with an initial controller and due to size of the problem. It should be noted that optimal network realizable controller problem in theorem 9 is actually designed benefiting coprime factorization of parallel plants: plant and its controller which is causing order inflation in the input to output map which is further affecting the orders of the obtained optimal controller. Another reason for order inflation is that optimal network realizable controller is function of three systems: K_0, $\bar{J}_1$ and Q.

7.8 Conclusion

In this chapter, we have shown an alternative way to parametrize all stabilizing network realizable controllers for networked systems. We have parametrized all stabilizing network realizable controllers as a function of an initial network realizable controller. Moreover, we have provided a necessary and sufficient network realizable controller problem for strongly connected networks. We have formulated a model matching problem and defined a necessary and sufficient optimal network realizable controller problem.

If the provided network is strongly connected, one can solve the provided optimal network realizable controller problem (7.22) after obtaining an initial network realizable controller benefiting theorem 12. Moreover, for any network, one can find an initial network realizable controller using theorem 5, then optimal network realizable controller can be obtained by solving proposed optimal network realizable controller problem in (7.22). Furthermore, one can benefit network distributed controller problems defined in [29], [43] to obtain an initial network realizable controller to solve the optimal network realizable controller problem defined in (7.22). After, obtaining optimal network realizable controller, we obtain a network implementable state-space realization using the network implementable state-space realization methodology formulated in section 4.2. A numerical example for strongly connected network has been provided in section 8.2 wherein comparisons with existing methods provided.

CHAPTER 8. NUMERICAL EXAMPLES

In this chapter we provide numerical examples for the optimal network realizable control problems defined in chapter 6 and 7. Moreover, network implementable state-space realization have been obtained using the network implementable state-space realization method defined in section 4.2.

8.1 Numerical Example - 1

As an example, 6-node system has been chosen with given A, B_w, B_u, C_z, C_y, D_{zw}, D_{zu}, D_{yw} as defined in equation (2.11) and this system's graph is as in figure 2.1. State space matrices are given as follows

$$A = \begin{bmatrix} 1.2 & -0.5 & 0.4 & -0.2 & 0 & 0 & 0.4 & -0.3 & 0 & 0 & 0 & 0 \\ 0.3 & 0.9 & -0.1 & 0.3 & 0 & 0 & -0.2 & 0.4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.1 & -0.3 & 0 & 0 & 0.2 & -0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & 1 & 0 & 0 & -0.2 & 0.3 & 0 & 0 & 0 & 0 \\ 0.4 & -0.2 & 0 & 0 & 0.8 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.1 & 0.3 & 0 & 0 & 0.3 & 1.2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.2 & -0.2 & 0 & 0 & 0 & 0 & 1.2 & -0.5 & 0.4 & -0.3 & 0 & 0 \\ -0.2 & 0.3 & 0 & 0 & 0 & 0 & 0.3 & 0.9 & -0.2 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.4 & -0.3 & 0 & 0 & 0.8 & -0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.2 & 0.4 & 0 & 0 & 0.3 & 1.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.4 & -0.2 & 1.1 & -0.3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.1 & 0.3 & 0.4 & 1 \end{bmatrix},$$

$$B_w = \begin{bmatrix} 0.3 & 0 & 0 & 0 & 0 & 0 \\ 0.2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.4 \\ 0 & 0 & 0 & 0 & 0 & 0.3 \end{bmatrix}, B_u = \begin{bmatrix} 1.2 & 0 & 0 & 0 & 0 & 0 \\ -0.8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -0.7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & -0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.1 & 0 & 0 \\ 0 & 0 & 0 & -0.7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.2 & 0 \\ 0 & 0 & 0 & 0 & -0.8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.1 \\ 0 & 0 & 0 & 0 & 0 & -0.7 \end{bmatrix},$$

$$C_z = \begin{bmatrix} 0.6 & 0.5 & 0.2 & 0.1 & 0 & 0 & 0.3 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.4 & 0 & 0 & 0.2 & 0.1 & 0 & 0 & 0 & 0 \\ 0.3 & 0.1 & 0 & 0 & 0.8 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.2 & 0.1 & 0 & 0 & 0 & 0 & 0.5 & 0.4 & 0.3 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.3 & 0.2 & 0 & 0 & 0.6 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.3 & 0.1 & 0.8 & 0.6 \end{bmatrix},$$

$$C_y = \begin{bmatrix} 0.4 & -0.5 & 0.2 & -0.3 & 0 & 0 & 0.3 & -0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & -0.6 & 0 & 0 & 0.1 & -0.1 & 0 & 0 & 0 & 0 \\ 0.2 & -0.3 & 0 & 0 & 0.6 & -0.4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.3 & -0.2 & 0 & 0 & 0 & 0 & 0.5 & -0.6 & 0.2 & -0.3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.1 & -0.1 & 0 & 0 & 0.4 & -0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.3 & -0.2 & 0.5 & -0.6 \end{bmatrix},$$

$$D_{zw} = \mathbf{blkdiag}(0.15, 0.25, 0.2, 0.2, 0.15, 0.25),$$

$$D_{zu} = \mathbf{blkdiag}(0.1, 0.2, 0.15, 0.1, 0.2, 0.15),$$

$$D_{yw} = \mathbf{blkdiag}(0.2, 0.25, 0.2, 0.1, 0.15, 0.2).$$

Problem (6.16) has been solved and let $\bar{W}_1^*$ and $\bar{W}_4^*$ be the solution of it. Q_4^* has been obtained as $\mathbf{vec}(S_4\bar{W}_4^*)^{-1}$ and optimal controller has been obtained using Q_4^* as $K^* = -Q_4^*(I - P_{22}Q_4^*)^{-1}$. Afterwards, we obtained network implementable state-space realization of the optimal controller, $\tilde{K}^*$, using network implementable state-space realizations $\tilde{V}$ and $\tilde{\bar{P}}_{22}$ as $\tilde{K}^* = \begin{bmatrix} 0 & I_{n_u} \end{bmatrix} \tilde{V}^*(I + \tilde{\bar{P}}_{22}\tilde{V}^*)^{-1} \begin{bmatrix} 0 & I_{n_y} \end{bmatrix}^T \in \mathfrak{S}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$ which is realized as in figure 4.1 where V and $\bar{P}_{22}$ is defined as in (4.2) and (3.3), respectively. $\tilde{\bar{P}}_{22}$ is obtained as given in (4.6) and $\tilde{V}$ has been obtained using network distributed realization method described in [19]. Order of attained optimal controller is 251 and the $\mathcal{H}_2$ cost has been re-calculated as 1.419. Moreover, sub-controllers order's are calculated as 59, 29, 49, 51, 56 and 7 for nodes $1-6$, respectively.

Moreover, we obtained an optimal network realizable controller using the method introduced in [43]. Realization of the obtained controller has been attained by both the realization method shown in [43] and the network state-space realization method shown in section 4.2 (see figure 4.1), it should be noted that for both of the realization methods, network implementable state space realization of stable systems have been obtained as described in section C.1 and illustrated in C.2. Corresponding controller orders has been calculated as 1021 and 454, respectively. One of the reasons of order difference is that while the network implementable realization method of [43] is based on network implementable realization of four stable systems, the network implementable realization method introduced in section 4.2 is based on network implementable realization of two stable systems.

Table 8.1 $\mathcal{H}_2$ bound results of 6-Node System

K^* Problem			
Problem	$\tilde{K}^*$ obtained by	$o(\tilde{K}^*)$	$\lvert\lvert T_{zw}\rvert\rvert_2^2$
-	-	12	1.259
[43]	[43]	1021	1.419
[43]	$\begin{bmatrix} 0 & I_{n_u} \end{bmatrix} \tilde{V}^*(I + \tilde{\bar{P}}_{22}\tilde{V}^*)^{-1} \begin{bmatrix} 0 & I_{n_y} \end{bmatrix}^T$	454	1.419
(6.16)	$\begin{bmatrix} 0 & I_{n_u} \end{bmatrix} \tilde{V}^*(I + \tilde{\bar{P}}_{22}\tilde{V}^*)^{-1} \begin{bmatrix} 0 & I_{n_y} \end{bmatrix}^T$	251	1.419

8.2 Numerical Example - 2

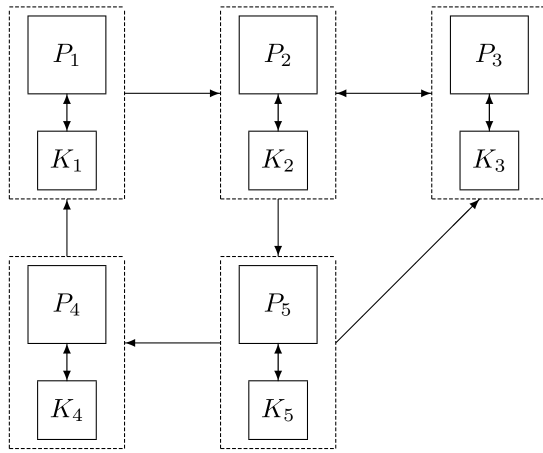

Figure 8.1 A strongly connected distributed system which consists of 5 sub-systems and their controller units interacting over a causal network.

As an example, 5-node system has been chosen with given A, B_w, B_u, C_z, C_y, D_{zw}, D_{zu}, D_{yw} as defined in Eq. (2.11) and this system's graph is as in figure 8.1. State space matrices are given as follows

$$A = \begin{bmatrix} 1.2 & -0.5 & 0 & 0 & 0 & 0 & 0.4 & -0.2 & 0 & 0 \\ 0.3 & 0.9 & 0 & 0 & 0 & 0 & -0.1 & 0.3 & 0 & 0 \\ 0.2 & -0.2 & 1.1 & -0.3 & 0.4 & -0.3 & 0 & 0 & 0 & 0 \\ -0.2 & 0.3 & 0.4 & 1 & -0.2 & 0.4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & -0.2 & 0.8 & -0.5 & 0 & 0 & 0.4 & -0.3 \\ 0 & 0 & -0.1 & 0.3 & 0.3 & 1.2 & 0 & 0 & -0.2 & 0.4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.2 & -0.5 & 0.2 & -0.2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.3 & 0.9 & -0.2 & 0.3 \\ 0 & 0 & 0.4 & -0.2 & 0 & 0 & 0 & 0 & 1.1 & -0.3 \\ 0 & 0 & -0.1 & 0.3 & 0 & 0 & 0 & 0 & 0.4 & 1 \end{bmatrix},$$

$$B_w = \begin{bmatrix} 0.3 & 0 & 0 & 0 & 0 \\ 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0.3 \\ 0 & 0 & 0 & 0 & 0.2 \end{bmatrix}, B_u = \begin{bmatrix} 1.2 & 0 & 0 & 0 & 0 \\ -0.8 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -0.7 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0 & 0 \\ 0 & 0 & -0.5 & 0 & 0 \\ 0 & 0 & 0 & 1.1 & 0 \\ 0 & 0 & 0 & -0.7 & 0 \\ 0 & 0 & 0 & 0 & 1.2 \\ 0 & 0 & 0 & 0 & -0.8 \end{bmatrix},$$

$$C_z = \begin{bmatrix} 0.6 & 0.5 & 0 & 0 & 0 & 0 & 0.2 & 0.1 & 0 & 0 \\ 0.3 & 0.1 & 0.5 & 0.4 & 0.3 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0.1 & 0.8 & 0.6 & 0 & 0 & 0.3 & 0.2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.6 & 0.5 & 0.3 & 0.1 \\ 0 & 0 & 0.2 & 0.1 & 0 & 0 & 0 & 0 & 0.5 & 0.4 \end{bmatrix},$$

$$C_y = \begin{bmatrix} 0.4 & -0.5 & 0 & 0 & 0 & 0 & 0.2 & -0.3 & 0 & 0 \\ 0.1 & -0.1 & 0.5 & -0.6 & 0.3 & -0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & -0.3 & 0.6 & -0.4 & 0 & 0 & 0.3 & -0.2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.4 & -0.5 & 0.1 & -0.1 \\ 0 & 0 & 0.2 & -0.3 & 0 & 0 & 0 & 0 & 0.5 & -0.6 \end{bmatrix},$$

$$D_{zw} = \textbf{blkdiag}(0.15, 0.15, 0.2, 0.2, 0.2),$$

$$D_{zu} = \textbf{blkdiag}(0.1, 0.2, 0.15, 0.1, 0.2),$$

$$D_{yw} = \textbf{blkdiag}(0.2, 0.25, 0.2, 0.1, 0.15).$$

We found a network realizable controller, $K_0 \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$ with the method introduced in theorem 12. Using this K_0, we solved the optimal network realizable controller problem given in (7.22) for W^*. After that, using optimal W^*, we obtained $Q^* = \textbf{vec}^{-1}(SW^*)$. We obtained the optimal network realizable controller by $K^* = \textbf{lft}(-K_0, \textbf{lft}(-\bar{J}_1, -Q^*))$ and network implementable state-space realization, $\tilde{K}^*$ is obtained as described in Method 2 in section 7.6.2. Obtained optimal controller order is 364. $\mathcal{H}_2$ cost has been recalculated with

$\tilde{K}^*$ and found as 1.316 and, each sub-controllers order's are calculated as 53, 100, 77, 64 and 70 for nodes 1, 2, 3, 4 and 5, respectively. In addition, we also obtained the network implementable optimal controller, $\tilde{K}^* \in \mathfrak{S}(\mathcal{G}, \mathcal{P}_{x_K}, \mathcal{P}_u, \mathcal{P}_y)$, by $\tilde{K}^* = \mathbf{lft}(-\tilde{K}_0, \mathbf{lft}(-\tilde{\bar{J}}_1, -\tilde{Q}^*))$ using network implementable state-space realizations $\tilde{K}_0$, $\tilde{\bar{J}}_1$ and $\tilde{Q}^*$. For this case, we obtained the order of $\tilde{K}^*$ as 691 and increase in optimal controller order, $o(\tilde{K}^*)$, is due to separate realization of K_0, $\bar{J}_1$ and Q^*. These two different optimal control realization indicates that obtaining realization of K^* as defined in Method 2 in section 7.6.2 yields a reduced controller order.

Moreover, we solved the sufficiency problem introduced in [43] to obtain optimal network distributed controller problem, and the network implementable state-space realization has been derived with both the method given in [43] and the block diagram in figure 4.1, and controller orders found as 1176 and 398, respectively. These results show that network implementable state-space realization method shown by figure 4.1 yields a smaller order. One of the main reason of this is that while the network realization method of [43] requires network realization of four stable systems, the network realization method as shown in figure 4.1 requires network realization of two stable systems.

Furthermore, in order to obtain an $\mathcal{H}_2$ optimal controller in infinite dimensional space, one can also solve the problem defined in (7.20) as a function of other initial network realizable controllers. A convex solution to $\mathcal{H}_2$ problem in (7.20) can be provided either with the vectorization method described in (7.22) or with the method described in [21]. For comparison, we also obtained initial network realizable controllers by solving problems described in [43] and [29], and network implementable state-space realizations, $\tilde{K}_0$, obtained by the network implementable state-space realization techniques described in the corresponding works. After solving optimal network realizable controller problem exists in (7.20), optimal controller realizations have been obtained with both using $\mathbf{lft}(-\tilde{K}_0, \mathbf{lft}(-\tilde{\bar{J}}_1, -\tilde{Q}^*))$ and the method described in Method 2 in section 7.6.2. Results can be observed on table 8.2.

We also obtained optimal network realizable controller using the optimal network realizable controller problem formulated in (6.16). After obtaining the network realizable optimal controller, we obtained the network implementable state-space realization using the network implementable state-space realization technique shown in section 4.2 and the order of network implementable controller has been found as 178. Results in table 8.2 shows the efficiency of our formulated method in terms of optimal controller order.

Table 8.2 $\mathcal{H}_2$ bound results of 5-Node System.

K_0 Problem			K^* Problem			
Pr.	Distr. S.S. real.	$o(\tilde{K}_0)$	Pr.	Distr. S.S. real.	$o(\tilde{K}^*)$	$\|\|T_{zw}\|\|_2^2$
-	-	-	-	-	10	1.134
-	-	-	[43]	[43]	1176	1.316
-	-	-	[43]	(4.7)	398	1.316
[29]	[29]	325	(7.20)	$-\mathbf{lft}(\tilde{K}_0, \mathbf{lft}(\tilde{\tilde{J}}_1, \tilde{Q}^*))$	1027	1.316
[43]	[43]	285	(7.20)	$-\mathbf{lft}(\tilde{K}_0, \mathbf{lft}(\tilde{\tilde{J}}_1, \tilde{Q}^*))$	625	1.316
(7.11)	(4.7)	202	(7.20)	$-\mathbf{lft}(\tilde{K}_0, \mathbf{lft}(\tilde{\tilde{J}}_1, \tilde{Q}^*))$	691	1.316
[29]	-	-	(7.20)	(4.7)	536	1.316
[43]	-	-	(7.20)	(4.7)	361	1.316
(7.11)	-	-	(7.20)	(4.7)	364	1.316
-	-	-	(6.16)	(4.7)	178	1.316

CHAPTER 9. MAIN RESULTS AND CONTRIBUTIONS

We formulated an all stabilizing controller parametrization benefiting a stably defined congruent plant and its all stabilizing controller parametrization which does not require to have doubly-coprime factorization of plants. Benefiting this parametrization, we defined a necessary and sufficient problem to obtain an output feedback controller and provided a two-step solution procedure for the formulated problem wherein a solution can be obtained in infinite dimensional space. Moreover, we obtained necessary and sufficient problems to attain a dynamic state-feedback controller and a dynamic state observer.

We provided a doubly coprime factorization of $\mathbf{blkdiag}(I, K)$ where K is a controller of the given plant. We showed that if the given controller inherits the network's sparsity and delay constraints of the given plant, then using the formulated coprime factorization of $\mathbf{blkdiag}(I, K)$, we showed that one can obtain network implementable state-realization of the given controller benefiting existing network implementable state-space realization method described for stable network realizable systems.

We formulated a necessary and sufficient network realizable controller problem for any network distributed system such that solution lies in infinite dimensional space which is one of the main contribution of this work. Furthermore, optimal network realizable controller problem has been formulated as a convex unconstrained problem which can be solved using existing solution techniques. Provided optimal network realizable controller problem does not require to have a doubly coprime factorization of plant or an initial controller. Obtained results with the provided numerical examples indicates that provided optimal network realizable controller problem and the presented network implementable realization method allow one to obtain optimal network implementable controller in fewer orders with respect to other existing problems and realization methods.

Furthermore, we provided an alternative way to parametrize all stabilizing network realizable controllers as a function of an initial network realizable controller benefiting existing Youla parametrization. We showed that for strongly connected networks if the plant is stabilizable and realizable, then there exists a network realizable controller in the form of delayed controllers. Moreover, we defined a necessary and sufficient optimal network realizable controller problem for network distributed systems as a function of an initial network realizable controller. Using network realizable controller in the form of delayed controller one can solve the provided optimal network realizable controller problem for strongly connected networks. Moreover, in the case of network is not strongly connected, one can benefit the network realizable controller problem provided in chapter 5 to have an initial network realizable controller to solve the optimal network realizable controller problem.

9.1 Directions for Future Work

As future research work, we would be interested in studying network implementable state-space realization of any given system which inherits the given graph's delay and sparsity constraints of the given network. Moreover, one can investigate the ways to obtain network implementable state-space realization of network realizable controllers in reduced order. We are also interested in solving network realizable control problems more efficiently without depending on the vectorization method. Moreover, for the larger sized networked systems, distributed solution methods can be investigated to obtain the network realizable controllers in a distributed fashion.

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APPENDIX A. ADDITIONAL PROOFS

In this section we will give proofs of some lemmas, theorems and corollaries which are not given in the previous chapters.

A.1 Proof of lemma 1

Proof. $\Rightarrow$ We will first show that for a stable Q, $K = -Q(I - P_{22}Q)^{-1}$ is an internally stabilizing controller of stable plant P_{22}.

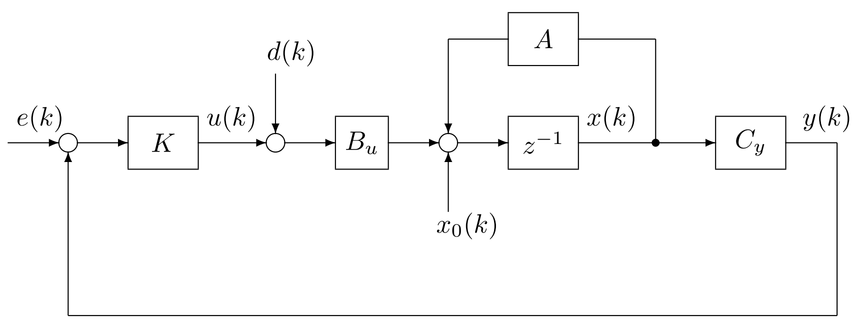

Figure A.1 Block diagram of P_{22} in feedback with controller K.

Let $x_0(k)$ be the noise of plant's states, $e(k)$ be the noise of output $u(k)$, and $d(k)$ be the noise of input $u(k)$ as in figure A.1. For an internally stabilizing controller, K, the input $\begin{bmatrix} x_0' & e' & d' \end{bmatrix}'$ to output $\begin{bmatrix} x' & y' & u' \end{bmatrix}'$ map needs to be stable. Let $Z = (zI - A - B_u K C_y)$. In a closed loop system, when all nine maps from $\begin{bmatrix} x_0' & e' & d' \end{bmatrix}'$ to $\begin{bmatrix} x' & y' & u' \end{bmatrix}'$:

$$\begin{bmatrix} Z^{-1} & Z^{-1}B_uK & Z^{-1}B_u \\ C_yZ^{-1} & P_{22}K(I-P_{22}K)^{-1} & P_{22}(I-KP_{22})^{-1} \\ KC_yZ^{-1} & K(I-P_{22}K)^{-1} & KP_{22}(I-KP_{22})^{-1} \end{bmatrix} \tag{A.1}$$

are stable, then closed loop can be called as internally stable. Let $Z_Q := (zI - A - B_uQC_y)$. For $K = -Q(I - P_{22}Q)^{-1}$, maps in (A.1) can be equivalently given as

$$\begin{bmatrix} (zI-A)^{-1}Z_Q(zI-A)^{-1} & -(zI-A)^{-1}B_uQ & (zI-A)^{-1}Z_Q(zI-A)^{-1}B_u \\ C_y(zI-A)^{-1}Z_Q(zI-A)^{-1} & -P_{22}Q & P_{22}(I-QP_{22}) \\ -QC_y(zI-A)^{-1} & -Q & -QP_{22} \end{bmatrix} \tag{A.2}$$

Since, the given plant P_{22} is stable, for a stable Q, nine maps given in (A.2) are stable. Therefore, for any stable Q, $K = -Q(I - P_{22}Q)^{-1}$ is an internally stabilizing controller of stable plant P_{22}.

$\Leftarrow$ Now, we will show that if the given K is a controller of P_{22}, then K can be parametrized as $K = -Q(I - P_{22}Q)^{-1}$ with a stable Q.

Let us define $Q := -K(I - P_{22}K)^{-1}$, then Q is stable since K is stabilizing P_{22}. Substituting $Q = -K(I - P_{22}K)^{-1}$ into $-Q(I - P_{22}Q)^{-1}$

$$\begin{aligned} -Q(I-P_{22}Q)^{-1} &= K(I-P_{22}K)^{-1}(I+P_{22}K(I-P_{22})^{-1})^{-1} \\ &= K \end{aligned} \tag{A.3}$$

which shows that with a stable Q, controller can be parametrized as $K = -Q(I - P_{22}Q)^{-1}$.

□

A.2 Proof of corollary 1

Proof. We will show that using (3.12c), (3.7b) and (3.7a), we can have the linear relationship between Q_1 and Q_2 as in (3.12a), then equation set in (3.4) will be equivalent to equation set in (3.13) and result will follow lemma 2.

Using (3.7b), we have

$$z^{-1}AB_uQ_4 = -(z^{-1}A - I)Q_2. \tag{A.4}$$

Moreover, multiplying (3.12c) from left with $z^{-1}AB_u$ brings in

$$z^{-1}AB_uQ_3(z^{-1}A - I) + z^{-1}AB_uQ_4(z^{-1}C_y) = 0. \tag{A.5}$$

Using (A.4), we can re-write (A.5) as follows

$$z^{-1}AB_uQ_3(z^{-1}A - I) - (z^{-1}A - I)Q_2(z^{-1}C_y) = 0. \tag{A.6}$$

So, using (A.6) one can obtain $z^{-1}AB_uQ_3$ as follows

$$z^{-1}AB_uQ_3 = (z^{-1}A - I)Q_2(z^{-1}C_y)(z^{-1}A - I)^{-1}. \tag{A.7}$$

Plugging in definition of $z^{-1}AB_uQ_3$ as in (A.7) into (3.7a) brings in

$$(z^{-1}A - I)Q_1 + (z^{-1}A - I)Q_2(z^{-1}C_y)(z^{-1}A - I)^{-1} = I. \tag{A.8}$$

Finally, we obtain the (3.12a) by multiplying (A.8) from right with $(z^{-1}A - I)$ and from left with $(z^{-1}A - I)^{-1}$. □

A.3 Proof of lemma 3

Proof. Existence of $\bar{Q}_1 \in \mathcal{RH}_\infty$ and $\bar{Q}_3 \in \mathcal{RH}_\infty$ which make the objective of (3.30) zero, equivalently means that there exists a stable right inverse of $X := \begin{bmatrix} z^{-1}A - I & z^{-1}B_u \end{bmatrix}$ which can be true if and only if $\text{rank}(X) = n_x$, for all $z \in \mathbb{C}$ with $|z| \geq 1$. For all $z \in \mathbb{C}$ with $|z| \geq 1$, rank of X can be equivalently find by rank of $\bar{X} := \begin{bmatrix} A - zI & B_u \end{bmatrix}$. Moreover, PBH test for stabilizability states that pair (A, B_u) is stabilizable if and only if $\bar{X}$ has rank n_x for all $z \in \mathbb{C}$ with $|z| \geq 1$. Therefore, we can conclude that pair (A, B_u) is stabilizable if and only if there exists a solution of (3.30) such that its objective is zero. □

A.4 Proof of lemma 4

Proof. Existence of $\bar{Q}_1 \in \mathcal{RH}_\infty$ and $\bar{Q}_2 \in \mathcal{RH}_\infty$, which makes the objective of the (3.31) zero, equivalently means that there exists a stable left inverse of $X = \begin{bmatrix} z^{-1}A - I \\ z^{-1}C_y \end{bmatrix}$ which can be true if and only if $\text{rank}(X) = n_x$, for all $z \in \mathbb{C}$ with $|z| \geq 1$. Moreover, rank of X can be equivalently found by rank of $Y = \begin{bmatrix} A - zI \\ C_y \end{bmatrix}$ for all $z \in \mathbb{C}$ with $|z| \geq 1$. PBH test for detectability states that pair (A, C_y) is detectable if and only if Y has rank n_x for all $z \in \mathbb{C}$ with $|z| \geq 1$ and result follows. □

A.5 Proof of corollary 4

Proof. Corollary 6 proves that there exists a zero norm solution to (3.38) if and only if there exists a zero norm solution to (3.35). Since (3.35) and (3.30), and (3.38) and (3.32) are same problems, by regarding corollary 6, it follows that there exists a zero norm solution to (3.30) if and only if there exists a zero norm solution to (3.32). Corollary 3 claims that given plant or the pair (A, B_u) is stabilizable if and only if there exists a zero norm solution to (3.30). Therefore, given plant or the pair (A, B_u) is stabilizable if and only if there exists a zero norm solution to (3.32). □

A.6 Proof of corollary 5

Proof. Corollary 7 proves that there exists a zero norm solution to (3.39) if and only if there exists a zero norm solution to (3.37). Since (3.37) and (3.31), and (3.39) and (3.33) are same problems, by regarding corollary 7, it follows that there exists a zero norm solution to (3.31) if and only if there exists a zero norm solution to (3.33). Corollary 4 claims that given plant or the pair (A, C_y) is detectable if and only if there exists a zero norm solution to (3.31). Therefore, given plant or the pair (A, C_y) is detectable if and only if there exists a zero norm solution to (3.33). □

A.7 Proof of lemma 5

Proof. Since $\bar{P}_f$ given in (3.34) is stable, dynamic state-feedback controllers of $\bar{P}_f$ can be parametrized as $\bar{F} = -\bar{Q}(I - \bar{P}_f\bar{Q})^{-1}$ with $\bar{Q} \in \mathcal{RH}_\infty$ according to lemma 1. In order to have a stabilizing F also for plant P_{22} with $C_y = I$, we need to find a $\bar{Q}$ which induces a structured $\bar{F} = \begin{bmatrix} I_{n_x} \\ F \end{bmatrix}$ such that $\mathbf{lft}(\bar{P}_f, \bar{F})$ is stable.

Using parametrization $\bar{F} = -\bar{Q}(I - \bar{P}_f\bar{Q})^{-1}$:

$$\begin{bmatrix} I_{n_x} \\ F \end{bmatrix} = -\begin{bmatrix} \bar{Q}_1 \\ \bar{Q}_3 \end{bmatrix} (I - \begin{bmatrix} z^{-1}A & z^{-1}B_u \end{bmatrix} \begin{bmatrix} \bar{Q}_1 \\ \bar{Q}_3 \end{bmatrix})^{-1} \tag{A.9}$$

Multiplying (A.9) from right with $(I - \begin{bmatrix} z^{-1}A & z^{-1}B_u \end{bmatrix} \begin{bmatrix} \bar{Q}_1 \\ \bar{Q}_3 \end{bmatrix})$ allows us to write the following equations

$$I - z^{-1}A\bar{Q}_1 - z^{-1}B_u\bar{Q}_3 = -\bar{Q}_1 \tag{A.10a}$$

$$F(I - z^{-1}A\bar{Q}_1 - z^{-1}B_u\bar{Q}_3) = -\bar{Q}_3 \tag{A.10b}$$

Using equation (A.10a) we obtain the equation

$$(z^{-1}A - I)\bar{Q}_1 + z^{-1}B_u\bar{Q}_3 = I \tag{A.11}$$

Moreover, using (A.10a), equation (A.10b) simplifies to

$$-F\bar{Q}_1 = -\bar{Q}_3 \tag{A.12}$$

Therefore, if there exist feasible $\bar{Q}_1 \in \mathcal{RH}_\infty$ and $\bar{Q}_3 \in \mathcal{RH}_\infty$ which satisfy (A.11), then we can obtain F as $F = \bar{Q}_3\bar{Q}_1^{-1}$. □

A.8 Proof of lemma 6

Proof. Since $\bar{P}_o$ given in (3.36) is stable, dynamic state observer of $\bar{P}_o$ can be parametrized as $\bar{L} = -(I - \bar{Q}\bar{P}_o)^{-1}\bar{Q}$ with $\bar{Q} \in \mathcal{RH}_\infty$ according to lemma 1. In order to have a stabilizing L also for plant P_{22} with $B_u = I$, we need to find a $\bar{Q}$ which induces a structured $\bar{L} = \begin{bmatrix} I_{n_x} & L \end{bmatrix}$ such that $\mathbf{lft}(\bar{P}_o, \bar{L})$ is stable.

Using parametrization $\bar{L} = -\bar{Q}(I - \bar{P}_o\bar{Q})^{-1}$:

$$\begin{bmatrix} I_{n_x} & L \end{bmatrix} = -(I - \begin{bmatrix} \bar{Q}_1 & \bar{Q}_2 \end{bmatrix} \begin{bmatrix} z^{-1}A \\ z^{-1}C_y \end{bmatrix})^{-1} \begin{bmatrix} \bar{Q}_1 & \bar{Q}_2 \end{bmatrix} \tag{A.13}$$

Multiplying (A.13) from left with $(I - \begin{bmatrix} \bar{Q}_1 & \bar{Q}_2 \end{bmatrix} \begin{bmatrix} z^{-1}A \\ z^{-1}C_y \end{bmatrix})$ allows us to write the following equations

$$I - z^{-1}\bar{Q}_1 A - z^{-1}\bar{Q}_2 C_y = -\bar{Q}_1 \tag{A.14a}$$

$$(I - z^{-1}\bar{Q}_1 A - z^{-1}\bar{Q}_2)L = -\bar{Q}_3 \tag{A.14b}$$

Using equation (A.14a) we obtain the equation

$$\bar{Q}_1(z^{-1}A - I) + z^{-1}\bar{Q}_2 B_u = I \tag{A.15}$$

Moreover, using (A.14a), equation (A.14b) simplifies to

$$-\bar{Q}_1 L = -\bar{Q}_2. \tag{A.16}$$

Therefore, if there exist feasible $\bar{Q}_1 \in \mathcal{RH}_\infty$ and $\bar{Q}_2 \in \mathcal{RH}_\infty$ which satisfy (A.15), then we can obtain L as $L = \bar{Q}_1^{-1}\bar{Q}_2$. □

A.9 Proof of corollary 6

Proof. For a given F, using parametrization $\bar{Q} = -\bar{F}(I - \bar{P}_f\bar{F})^{-1}$, we have that $\bar{Q}_1 = -zZ^{-1}$ and $\bar{Q}_3 = -zKZ^{-1}$ where $Z = zI - A - B_uF$. Therefore, $Q_1 \in \mathcal{RH}_\infty$ satisfying (A.11) admits a form $\bar{Q}_1 := -I + z^{-1}\hat{Q}_1$ wherein $\hat{Q}_1$ is stable and casual, so we can re-write (A.11) as follows

$$(z^{-1}A - I)(-I + z^{-1}\hat{Q}_1) + z^{-1}B_u\bar{Q}_3 = I. \tag{A.17}$$

which simplifies to

$$-A + (z^{-1}A - I)\hat{Q}_1 + B_u\bar{Q}_3 = 0. \tag{A.18}$$

A feasible solution to (A.18) must satisfy the following

$$-LA + L(z^{-1}A - I)\hat{Q}_1 = 0. \tag{A.19}$$

Therefore, there exists a state-feedback controller if and only if there exists a feasible solution to (A.19) such that $\hat{Q}_1 \in \mathcal{RH}_\infty$, which equivalently makes the objective of (3.38) zero. Let $\hat{Q}_1^*$ be a feasible solution to (A.19), then using (A.18), $\bar{Q}_3^*$ can be recovered as $B_u^\dagger(A - (z^{-1}A - I)\hat{Q}_1^*)$. Then, a dynamic state controller can be synthesized as $F = \bar{Q}_3(z^{-1}\hat{Q}_1 - I)^{-1}$ by regarding lemma 5. □

A.10 Proof of corollary 7

Proof. For a given L, using parametrization $\bar{Q} = -(I - \bar{L}\bar{P}_o)^{-1}\bar{L}$, we have that $Q_1 = -zZ^{-1}$ and $Q_2 = -zZ^{-1}L$ where $Z = zI - A - LC_y$. Therefore, $Q_1 \in \mathcal{RH}_\infty$ satisfying (A.15) admits a form $\bar{Q}_1 = -I + z^{-1}\hat{Q}_1$, so we can re-write (A.15) as follows

$$(z^{-1}A - I)(-I + z^{-1}\hat{Q}_1) + z^{-1}\bar{Q}_2C_y = I. \tag{A.20}$$

which simplifies to

$$-A + (z^{-1}A - I)\hat{Q}_1 + \bar{Q}_2C_y = 0. \tag{A.21}$$

A feasible solution to (A.21) must satisfy the following

$$-AR + \hat{Q}_1(z^{-1}A - I)R = 0. \tag{A.22}$$

Therefore, there exists a state observer if and only if there exists a feasible solution to (A.22) such that $\hat{Q}_1 \in \mathcal{RH}_\infty$, which equivalently makes the objective of (3.39) zero. Let $\hat{Q}_1^*$ be a feasible solution to (A.22), then using (A.21), $\bar{Q}_2$ can be recovered as $(A-\hat{Q}_1(z^{-1}A-I))C_y^{\dagger}$. Then, a dynamic state observer can be synthesized as $L = (z^{-1}\hat{Q}_1^* - I)^{-1}\bar{Q}_2^*$ by regarding lemma 6. □

A.11 Proof of lemma 7

Proof. As shown in proof of 2, one can parametrize $\mathbf{blkdiag}(I_{n_x}, K)$ as $-\bar{Q}(I - \bar{P}_{22}\bar{Q})^{-1}$ where $\bar{Q} \in \mathcal{RH}_\infty$ satisfies the equalities in (3.4). By defining $V := -\bar{Q}$ and $W := (I-\bar{P}_{22}\bar{Q})$, we can express $\bar{K} = -\bar{Q}(I-P_{22}\bar{Q})$ as VW^{-1}. To be able to define V, we will find definition of $\bar{Q}$ in terms of K which satisfies equations (3.4).

As it is given in (3.10), we can express Q_4 as $Q_4 = -K(I - P_{22}K)^{-1}$. Moreover, using (3.12c) and $Q_4 = -K(I - P_{22}K)^{-1}$, we can express Q_3 in terms of K using as in the followings

$$\begin{aligned} Q_3 &= Q_4 C_y (zI - A)^{-1} \\ &= -K(I - C_y(zI - A)^{-1}B_u K)^{-1} C_y (zI - A)^{-1} \\ &= -KC_y(I - (zI - A)^{-1}B_u K C_y)^{-1}(zI - A)^{-1} \\ &= -KC_y(zI - A - B_u K C_y)^{-1} \end{aligned} \tag{A.23}$$

Moreover, by plugging the definition of Q_3 obtained in (A.23) into (3.7a), we can express Q_1 as follows

$$\begin{aligned} Q_1 &= (z^{-1}A - I)^{-1} - z^{-1}(z^{-1}A - I)^{-1} A B_u Q_3 \\ &= (z^{-1}A - I)^{-1} + z^{-1}(z^{-1}A - I)^{-1} A B_u K C_y (zI - A - B_u K C_y)^{-1} \\ &= -I - A(zI - A - B_u K C_y)^{-1} \end{aligned} \tag{A.24}$$

Furthermore, using (3.7b) and $Q_4 = -K(I - P_{22}K)^{-1}$, we can express Q_2 in terms of K as in the following

$$
\begin{aligned}
Q_2 &= (zI - A)^{-1} A B_u Q_4 \\
&= -(zI - A)^{-1} A B_u K (I - C_y (zI - A)^{-1} B_u K)^{-1} \\
&= -A(zI - A)^{-1} (I - B_u K C_y (zI - A)^{-1})^{-1} B_u K \\
&= -A(zI - A - B_u K C_y)^{-1} B_u K
\end{aligned}
\tag{A.25}
$$

Using (3.10), (A.23), (A.24) and (A.25), we can express Q as follows

$$
\bar{Q} = \begin{bmatrix} -I - AZ^{-1} & -AZ^{-1} B_u K \\ -K C_y Z^{-1} & -K(I - PK)^{-1} \end{bmatrix}. \tag{A.26}
$$

Therefore, using definition of $V = -\bar{Q}$, we obtain V as given in (4.2). Also, using definition of $W = (I - \bar{P}_{22}\bar{Q})$, we obtain W as given in (4.2). Instead of showing $W = (I - \bar{P}_{22}\bar{Q})$ is equivalent to definition of W given in (4.2), we shortly show that W defined as in (4.2) satisfies the equality $V = \bar{K}W$:

$$
\begin{aligned}
V &= \bar{K}W \\
&= \begin{bmatrix} I & 0 \\ 0 & K \end{bmatrix} \begin{bmatrix} I + AZ^{-1} & AZ^{-1} B_u K \\ C_y Z^{-1} & (I - P_{22} K)^{-1} \end{bmatrix} \\
&= \begin{bmatrix} I + AZ^{-1} & AZ^{-1} B_u K \\ K C_y Z^{-1} & K(I - P_{22} K)^{-1} \end{bmatrix}
\end{aligned}
\tag{A.27}
$$

Moreover, we can equivalently parametrize $\bar{K}$ as

$$
\bar{K} = -(I - \bar{Q}\bar{P}_{22})^{-1} \bar{Q} \tag{A.28}
$$

Let us define $\bar{V} := -\bar{Q}$, so we obtain the $\bar{V}$ defined in (4.2). Also, define $\bar{W} := (I - \bar{Q}\bar{P}_{22})$ which is equivalent to $\bar{W}$ given in (4.2). Instead of showing $\bar{W} = (I - \bar{Q}\bar{P}_{22})$ is equivalent to $\bar{W}$ in (4.2), we shortly show that $\bar{W}$ defined as in (4.2) satisfies the equality $\bar{V} = \bar{W}\bar{K}$:

$$\begin{aligned}\bar{V} &= \bar{W}\bar{K} \\ &= \begin{bmatrix} I + AZ^{-1} & AZ^{-1}B_u \\ KC_yZ^{-1} & (I - KP_{22})^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & K \end{bmatrix} \\ &= \begin{bmatrix} I + AZ^{-1} & AZ^{-1}B_uK \\ KC_yZ^{-1} & (I - KP_{22})^{-1}K \end{bmatrix}\end{aligned} \tag{A.29}$$

Next, we will show that equality (4.3) holds. Let Φ be defined as $\Phi := \begin{bmatrix} \Phi_1 & \Phi_2 \\ \Phi_3 & \Phi_4 \end{bmatrix}$. Then, we have

$$\begin{bmatrix} \Phi_1 & \Phi_2 \\ \Phi_3 & \Phi_4 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{A.30}$$

1. First, we will show that $\Phi_1 = \bar{X}W - \bar{Y}V = I$ holds.

$$\Phi_1 := \begin{bmatrix} \Phi_1^1 & \Phi_1^2 \\ \Phi_1^3 & \Phi_1^4 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{A.31}$$

$$\begin{aligned}\Phi_1^1 &= I + AZ^{-1} - z^{-1}A - z^{-1}A^2Z^{-1} - z^{-1}AB_uKC_yZ^{-1} \\ &= (Z + A - z^{-1}AZ - z^{-1}A^2)Z^{-1} \\ &= ZZ^{-1} \\ &= I.\end{aligned}$$

Using equality $K(I - PK)^{-1} = K + KC_yZ^{-1}B_uK$, one can obtain the followings

$$\begin{aligned}\Phi_1^2 &= AZ^{-1}B_uK - z^{-1}A^2Z^{-1}B_uK - z^{-1}AB_uK(I - P_{22}K)^{-1} \\ &= (A - z^{-1}A^2 - z^{-1}AZ - z^{-1}AB_uKC_y)Z^{-1}B_uK \\ &= 0.\end{aligned}$$

$$\begin{aligned}\Phi_1^3 &= C_yZ^{-1} - z^{-1}C_y - z^{-1}C_yAZ^{-1} - z^{-1}C_yB_uKC_yZ^{-1} \\ &= (C_y - z^{-1}C_yZ - z^{-1}C_yA - z^{-1}C_yB_uKC_y)Z^{-1} \\ &= 0.\end{aligned}$$

Using equality $AZ^{-1}B_uK = A(zI-A)^{-1}B_uK(I-P_{22}K)^{-1}$, one can obtain the following equalities

$$\begin{aligned}\Phi_1^4 &= (I-PK)^{-1} - z^{-1}C_y(AZ^{-1}B_uK + B_uK(I-P_{22}K)^{-1})\\ &= \left[I - z^{-1}C_yA(zI-A)^{-1}B_uK - z^{-1}C_yB_uK\right](I-P_{22}K)^{-1}\\ &= \left[I - z^{-1}C_y(zI-A)^{-1}(AB_uK + (zI-A)B_uK)\right](I-P_{22}K)^{-1}\\ &= \left[I - C_y(zI-A)^{-1}B_uK\right](I-P_{22}K)^{-1}\\ &= I.\end{aligned}$$

2. $\Phi_2 = \bar{X}Y - \bar{Y}X = 0$ is an immediate result of $X = \bar{X}$ and $Y = \bar{Y}$.

3.Equalities $\bar{K}W = V$ and $\bar{W}\bar{K} = \bar{V}$ can be observed trivially, which yield $VW^{-1} = \bar{W}^{-1}\bar{V}$ and therefore, we have $\Phi_3 = \bar{V}W - \bar{W}V = 0$.

4. Now, we will prove the equality $\Phi_4 = -\bar{V}Y + \bar{W}X = I$.

$$\Phi_4 := \begin{bmatrix}\Phi_4^1 & \Phi_4^2\\ \Phi_4^3 & \Phi_4^4\end{bmatrix} = \begin{bmatrix}I & 0\\ 0 & I\end{bmatrix} \tag{A.32}$$

$$\begin{aligned}\Phi_4^1 &= -z^{-1}A - z^{-1}AZ^{-1}A - z^{-1}AZ^{-1}B_uKC_y + I + AZ^{-1}\\ &= -z^{-1}AZ^{-1}(z^{-1}I - A - B_uKC_y) - z^{-1}A + I\\ &= I.\end{aligned}$$

Using equality $K(I-P_{22}K)^{-1} = K + KC_yZ^{-1}B_uK$, one can obtain the followings

$$\begin{aligned}\Phi_4^2 &= -z^{-1}(I + AZ^{-1})AB_u - z^{-1}AZ^{-1}B_uKC_yB_u + AZ^{-1}B_u\\ &= -z^{-1}AB_u + z^{-1}AZ^{-1}(zI - A - B_uKC_y)B_u\\ &= 0.\end{aligned}$$

$$\begin{aligned}\Phi_4^3 &= -\,z^{-1}KC_yZ^{-1}A - z^{-1}K(I-P_{22}K)^{-1}C_y + KC_yZ^{-1}\\ &= -\,z^{-1}KC_yZ^{-1}A - z^{-1}(K + KC_yZ^{-1}B_uK)C_y + KC_yZ^{-1}\\ &= z^{-1}KC_yZ^{-1}(zI - A - B_uKC_y) - z^{-1}KC_y\\ &= 0.\end{aligned}$$

Using equality $K(I-P_{22}K)^{-1} = K + KC_yZ^{-1}B_uK$ and $(I-KP_{22})^{-1} = I + KC_y(zI - A - B_uKC_y)^{-1}B_u$, one can obtain the followings

$$\begin{aligned}
\Phi_4^4 =& -z^{-1}KC_yZ^{-1}AB_u - z^{-1}K(I-P_{22}K)^{-1}C_yB_u + (I-KP_{22})^{-1} \\
=& -z^{-1}KC_yZ^{-1}AB_u - z^{-1}(K + KC_yZ^{-1}B_uK)C_yB_u + I \\
&+ KC_y(zI - A - B_uKC_y)^{-1}B_u \\
=& z^{-1}KZ^{-1}(zI - A - B_uKC_y)B_u - z^{-1}KC_yB_u + I \\
=& I.
\end{aligned}$$

These derivations proves that $\Phi = I$ is satisfied.

Moreover, since $\bar{Q}$ is stable, V and $\bar{V}$ are stable by their definition. Moreover, since $\bar{P}_{22}$ is stable, then $W = I - \bar{P}_{22}\bar{Q}$ and $\bar{W} = I - \bar{Q}\bar{P}_{22}$ are also stable. Furthermore, we have shown that (4.3) holds. Therefore, if K stabilizes P_{22}, then a doubly coprime factorization of $\bar{K} = \mathbf{blkdiag}(I_{n_x}, K)$ can be given as $\bar{K} = VW^{-1} = \bar{W}^{-1}\bar{V}$. $\square$

APPENDIX B. EXTENSIONS

B.1 Controllers of Continuous Time Systems

For a given continuous time system $\dot{x}(t) = Ax(t) + B_u u(t)$, for a sampling period of T, its discrete time system can be given as $x(k+1) = A_d x(k) + B_d u(k)$ where

$$A_d = e^{AT} = I + AT + \frac{1}{2!}A^2T^2 + \frac{1}{3!}A^3T^3 + ...$$

$$B_d = \int_0^T e^{A\tau}B_u d\tau = B_uT + \frac{1}{2!}AB_uT^2 + \frac{1}{3!}A^2B_uT^3 + ...$$

For a given structured $A \in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_x, \mathcal{P}_y)$ its structure does not preserve for the terms A^m for $m > 1$, therefore discretization does not preserve the network structure which should be avoided for the networked structured systems.

To obtain all stabilizing controllers of continuous time systems, $\bar{P}_{22}$ and $\bar{K}$ needs to be chosen different than $\bar{P}_{22}$ defined in (3.3). One may choose

$$\bar{P}_{22} = \begin{bmatrix} (s+a)^{-1}A & (s+a)^{-1}AB_u \\ (s+a)^{-1}C_y & (s+a)^{-1}C_yB_u \end{bmatrix}$$

with $\bar{K} = \mathbf{blkdiag}(s^{-1}(s+a)I, s^{-1}(s+a)K)$ wherein a is any $a \in \mathbb{R}^+$ and then follow the proof of lemma 2 to obtain all stabilizing controllers for continuous time systems.

Moreover, on $\mathcal{H}_2$ optimal control problem of continuous time systems, closed loop map needs to be strictly proper, i.e. $D_{zw} + D_{zu}D_KD_{yw} = 0$.

B.2 Output Feedback Controller Problems for Special Cases

In this section we will provide controller problems for the cases of "null space of $(AB_u)^T$ is empty", "null space of C_y is empty" and "null spaces of both $(AB_u)^T$ and C_y are empty".

Corollary 10. *Let the plant be as given in (2.11). Let AB_u be such that null space of $(AB_u)^T$ is empty and let C_y be such that null space of C_y is not empty. Let R be a concatenation of null space vectors of C_y. There exists an internally stabilizing controller, if and only if there exists a $\tilde{Q}_1$ which makes the objective of the following problem zero.*

$$\begin{aligned} \min_{\tilde{Q}_1} \quad & \left\| -A^2 R + \tilde{Q}_1(z^{-1}A - I)R \right\|_2^2 \\ s.t. \quad & \tilde{Q}_1 \in \mathcal{RH}_\infty \end{aligned} \tag{B.1}$$

Moreover, let $\tilde{Q}_1^*$ be a solution to (B.1) such that objective of (B.1) is zero, and let $Q_4^* = (AB_u)^{-1}(-A^2(z^{-1}A - I) + (z^{-1}A - I)\tilde{Q}_1^*(z^{-1}A - I))C_y^\dagger$, then an internally stabilizing can be given as $K = -Q_4^*(I - P_{22}Q_4^*)^{-1}$.

Proof. Proof follows corollary 3. □

Problem (B.1) is in the form of $\mathcal{H}_2$ problem and existing solution methods of $\mathcal{H}_2$ problem can be used to have a solution in infinite dimensional space.

Corollary 11. *Let the plant be as given in (2.11). Let AB_u be such that null space of $(AB_u)^T$ is not empty and let C_y be such that null space of C_y is empty. Let L_T be a concatenation of null space vectors of $(AB_u)^T$ and define $L := L_T^T$. There exists an internally stabilizing controller, if and only if there exists a $\tilde{Q}_1$ which makes the objective of the following problem zero.*

$$\begin{aligned} \min_{\tilde{Q}_1} \quad & \left\| -LA^2 + L(z^{-1}A - I)\tilde{Q}_1 \right\|_2^2 \\ s.t. \quad & \tilde{Q}_1 \in \mathcal{RH}_\infty \end{aligned} \tag{B.2}$$

Moreover, let $\tilde{Q}_1^*$ be a solution to (B.2) such that objective of (B.2) is zero, and let $Q_4^* = (AB_u)^\dagger(-A^2(z^{-1}A - I) + (z^{-1}A - I)\tilde{Q}_1^*(z^{-1}A - I))C_y^{-1}$, then an internally stabilizing can be given as $K = -Q_4^*(I - P_{22}Q_4^*)^{-1}$.

Proof. Proof follows corollary 3. □

Problem (B.2) can be solved as a classical $\mathcal{H}_2$ problem to have a solution in infinite dimensional space.

Corollary 12. *Let the plant be as given in* (2.11)*. Let* A*,* B_u *and* C_y *be such that null spaces of both* $(AB_u)^T$ *and* C_y *are empty. Let* $\tilde{Q}_1$ *be any system in space* $\mathcal{RH}_\infty^{n_x \times n_x}$*. Let* $Q_4 = (AB_u)^{-1}(-A^2(z^{-1}A-I)+(z^{-1}A-I)\tilde{Q}_1(z^{-1}A-I))C_y^{-1}$*, then an internally stabilizing controller can be given as* $K = -Q_4(I - P_{22}Q_4)^{-1}$*.*

Proof. Proof follows corollary 3 □

By regarding corollary 12, one can trivially give a stabilizing controller for plant P as $K = -Q_4(I - P_{22}Q_4)^{-1}$ wherein $Q_4 = -(AB_u)^{-1}A^2(z^{-1}A-I)C_y^{-1}$ when A, B_u and C_y are such that null spaces of both $(AB_u)^T$ and C_y are empty.

B.3 Alternative Controller Problems

Controller problems given in chapter 3 whether requires to use vectorization technique or two-step solution procedure as in given in theorem 3 to attain a solution in infinite dimensional space. In this section, we propose alternative methods to obtain a controller in one step. While the problems given in chapter 3 are necessary and sufficient problems to obtain a controller, problems given in this chapters are sufficient problems. If there exists a solutions to the problems given in this chapter then one can obtain a controller for the given plant.

B.3.1 Alternative Controller Problem - 1

In this section we propose an alternative problem to solve the controller synthesis problem in one step.

Corollary 13. *Let the plant be as given in (2.11). Let* $AB_u = U_B S_B V_B^T$ *and* $C_y = U_C S_C V_C^T$ *be singular value decompositions of* AB_u *and* C_y*, such that* $S_B = \begin{bmatrix} \bar{S}_B \\ 0_{n_x - n_u, n_u} \end{bmatrix}$ *and* $S_C = \begin{bmatrix} \bar{S}_C & 0_{n_y, n_x - n_y} \end{bmatrix}$*. Let* $\Theta(\tilde{Q}_1)$ *be defined as*

$$\Theta(\tilde{Q}_1) = -A^2(z^{-1}A - I) + (z^{-1}A - I)\tilde{Q}_1(z^{-1}A - I). \tag{B.3}$$

Let $\Omega(\tilde{Q}_1)$ *be defined as follows*

$$\Omega(\tilde{Q}_1) := \begin{bmatrix} \Omega_1(\tilde{Q}_1) & \Omega_2(\tilde{Q}_1) \\ \Omega_3(\tilde{Q}_1) & \Omega_4(\tilde{Q}_1) \end{bmatrix} := U_B^{-1}\Theta(\tilde{Q}_1)(V_C^T)^{-1} \tag{B.4}$$

where $\Omega_1(\tilde{Q}_1) \in \mathcal{RH}_\infty^{n_u \times n_y}$ *for a* $\tilde{Q}_1 \in \mathcal{RH}_\infty$*. There exists an internally stabilizing controller, if and only if there exists a* $\tilde{Q}_1$ *which makes the objective of following problem zero.*

$$\begin{aligned} \min_{\tilde{Q}_1} \quad & \sum_{i=2}^{4} \left\| \Omega_i(\tilde{Q}_1) \right\|_2^2 \\ s.t. \quad & \tilde{Q}_1 \in \mathcal{RH}_\infty \end{aligned} \tag{B.5}$$

Moreover, let $\tilde{Q}_1^*$ *be a solution to (B.5), and* $Q_4^* = (AB_u)^\dagger \Theta(\tilde{Q}_1^*) C_y^\dagger$*, then, an internally stabilizing can be given as* $K = -Q_4^*(I - P_{22}Q_4^*)^{-1}$*.*

Proof. If we multiply (3.17) from left with U_A^{-1} and from right with V_C^{-1}, we obtain the following

$$\begin{aligned} & U_B^{-1}\left[-A^2(z^{-1}A - I) + (z^{-1}A - I)\tilde{Q}_1(z^{-1}A - I)\right]V_C^{-1} \\ & = U_B^{-1}AB_u Q_4 C_y V_C^{-1} = \begin{bmatrix} \Omega_1 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{B.6}$$

Since for a $\tilde{Q}_1 \in \mathcal{RH}_\infty$, Ω_1 is in space $\mathcal{RH}_\infty$, we need to only solve for zero equalities in (B.6). Therefore, constraint given in (3.17) can be equivalently written with the constraints set: $\Omega_2(\tilde{Q}_1) = 0$, $\Omega_3(\tilde{Q}_1) = 0$ and $\Omega_4(\tilde{Q}_1) = 0$, and results follow corollary 2. □

In order to be able use the existing $\mathcal{H}_2$ solution methods, minimized function need to be affine in variable, i.e. in the form of $H + UQV$ for some H, U, V wherein Q is variable. As it can be noticed, problem given in corollary 13 is not in this form, hence, classical solution methods can not be applied. Therefore, one may take advantage of the vectorization method as shown in [37] to attain an equivalent of minimized function where existing solution methods of $\mathcal{H}_2$ problem can be applied.

Next, we propose an alternative problem to solve an internally stabilizing controller problem by taking equation (B.6) into consideration to find an internally stabilizing controller by avoiding vectorization method.

Multiplying left hand side of (B.6) from left with I_u and from right with I_y brings in

$$\begin{aligned} & I_u U_A^{-1}\left[(z^{-1}A - I)\tilde{Q}_1(z^{-1}A - I) - A^2(z^{-1}A - I)\right] V_C^{-1} I_y \\ & = \begin{bmatrix} \epsilon_1^2 \Omega_1(\tilde{Q}_1) & \epsilon_1 \Omega_2(\tilde{Q}_1) \\ \epsilon_1 \Omega_3(\tilde{Q}_1) & \Omega_4(\tilde{Q}_1) \end{bmatrix}. \end{aligned} \tag{B.7}$$

If we put equation (B.7) through minimization, $\epsilon_1^2 \left\| \Omega_1(\tilde{Q}_1) \right\|_2^2$ possesses a negligible magnitude for a small enough $\epsilon_1 \in \mathbb{R}^+$. Therefore, one can subject (B.7) to norm minimization in search of a controller.

Proposition 1. *Let the plant be as given in (2.11). Let $\epsilon_1 \in \mathbb{R}^+$ be small enough. Let $AB_u = U_B S_B V_B^T$ and $C_y = U_C S_C V_C^T$ be singular value decompositions of AB_u and C_y, such that $S_B = \begin{bmatrix} \bar{S}_B \\ 0_{n_x - n_u, n_u} \end{bmatrix}$ and $S_C = \begin{bmatrix} \bar{S}_C & 0_{n_y, n_x - n_y} \end{bmatrix}$. Let $I_u = \textbf{blkdiag}(\epsilon_1 I_{n_u}, I_{n_x - n_u})$ and $I_y = \textbf{blkdiag}(\epsilon_1 I_{n_y}, I_{n_x - n_y})$. Let $H_\Omega = I_u U_B^{-1}(-A^2(z^{-1}A - I))V_C^{-1} I_y$, $U_\Omega = I_u U_B^{-1}(z^{-1}A - I)$ and $V_\Omega = (z^{-1}A - I)V_C^{-1} I_y$. Let $\tilde{Q}_1^*$ be a solution of the following problem*

$$\begin{aligned} \min_{\tilde{Q}_1} \quad & \left\| H_\Omega + U_\Omega \tilde{Q}_1 V_\Omega \right\|_2^2 \\ s.t. \quad & \tilde{Q}_1 \in \mathcal{RH}_\infty. \end{aligned} \tag{B.8}$$

Let $Q_4^* = (AB_u)^\dagger \Theta(\tilde{Q}_1^*) C_y^\dagger$ where $\Theta(\tilde{Q}_1^*)$ defined as in (B.3). If $\tilde{Q}_1^*$ makes the constraint $\sum_{i=2}^4 \left\| \Omega_i(\tilde{Q}_1) \right\|_2^2 < \epsilon_2$ satisfied, where $\Omega_2(\tilde{Q}_1)$, $\Omega_3(\tilde{Q}_1)$ and $\Omega_4(\tilde{Q}_1)$ are as defined in (B.4) *for a negligibly small* $\epsilon_2 \in \mathbb{R}^+$*, then an internally stabilizing controller can be given as* $K = -Q_4^*(I - P_{22}Q_4^*)^{-1}$.

Proof. Proof follows corollary 3. □

Problem defined in (B.8) is an unconstrained $\mathcal{H}_2$ problem which can be solved using standard techniques to find the solution. If the results of the problem (B.8) satisfies $\sum_{i=2}^4 \left\|\Omega_i(\tilde{Q}_1)\right\|_2^2 < \epsilon_2$ for a negligibly small ϵ_2, than one can attain a controller.

B.3.2 Alternative Controller Problem - 2

In this section, we will propose another relaxed method to find stabilizing controller to solve the control synthesis problem in one step by avoiding vectorization method.

Corollary 14. *Let the plant be as given in* (2.11)*. Let* $\Phi(\bar{Q})$ *be defined as follows*

$$\Phi(\bar{Q}) = \begin{bmatrix} z^{-1}A - I & 0 \\ 0 & Q_4 \end{bmatrix} - \begin{bmatrix} z^{-1}A - I & z^{-1}AB_u \\ 0 & I \end{bmatrix} \bar{Q} \begin{bmatrix} z^{-1}A - I & 0 \\ z^{-1}C_y & I \end{bmatrix}$$

There exists an internally stabilizing controller, $K = -Q_4(I - P_{22}Q_4)^{-1}$*, if and only if there exists* $\bar{Q} := \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix}$ *which makes the objective of following problem zero.*

$$\begin{aligned} \min_{\bar{Q}} \quad & \left\|\Phi(\bar{Q})\right\|_2^2 \\ s.t. \quad & \bar{Q} \in \mathcal{RH}_\infty. \end{aligned} \tag{B.9}$$

Proof. There exists a controller if and only if there exists a feasible $\bar{Q} \in \mathcal{RH}_\infty$ which satisfies the equality constraints given in (3.4) according to lemma 2. Multiplying (3.4a) from left with $\begin{bmatrix} z^{-1}A - I \\ z^{-1}C_y \end{bmatrix}$ yields

$$\begin{bmatrix} z^{-1}A - I & z^{-1}AB_u \end{bmatrix} \bar{Q} \begin{bmatrix} z^{-1}A - I \\ z^{-1}C_y \end{bmatrix} = z^{-1}A - I. \tag{B.10}$$

Moreover, multiplying (3.4b) from right with $\begin{bmatrix} z^{-1}A - I & z^{-1}AB_u \end{bmatrix}$ also yields equation (B.10).

After multiplying (3.7b) with $z^{-1}C_y$ from right, we subtract it from (B.10) and obtain the following

$$(z^{-1}A - I)Q_1(z^{-1}A - I) + z^{-1}AB_uQ_3(z^{-1}A - I) = z^{-1}A - I. \tag{B.11}$$

Multiplying (B.11) with $(z^{-1}A - I)^{-1}$ from right brings in (3.7a). Therefore, equation constraint pair (3.7b) and (3.7a) can be replaced with (3.7b) and (B.10).

Moreover, after multiplying (3.12c) with $z^{-1}AB_u$ from left, we subtract it from (B.10) and obtain the following

$$(z^{-1}A - I)Q_1(z^{-1}A - I) + z^{-1}(z^{-1}A - I)Q_2C_y = z^{-1}A - I. \tag{B.12}$$

Multiplying (B.12) with $(z^{-1}A - I)^{-1}$ from left brings in (3.12a). Therefore, equation constraint pair (3.12c) and (3.12a) can be replaced with (3.12c) and (B.10). Therefore, constraint set (3.12c), (3.7b), (3.7a) and (3.12a) can be equivalently solved with (3.12c), (3.7b) and (B.10). Moreover, one can write constraint set (3.12c), (3.7b) and (B.10) as in the following

$$\begin{bmatrix} z^{-1}A - I & z^{-1}AB_u \\ 0 & I \end{bmatrix} \bar{Q} \begin{bmatrix} z^{-1}A - I & 0 \\ z^{-1}C_y & I \end{bmatrix} = \begin{bmatrix} z^{-1}A - I & 0 \\ 0 & Q_4 \end{bmatrix} \tag{B.13}$$

Therefore, a feasible $\bar{Q} \in \mathcal{RH}_\infty$ which satisfies (B.13) is a feasible solution of (3.4) which can be equivalently found by problem B.9 if there exists any, therefore results follow lemma 2. □

Problem in (B.9) is in the form $H(Q_4) + U\bar{Q}V$, to be able to solve the problem using existing solution methods of $\mathcal{H}_2$ problem, we need also a H which does not depend on Q_4. Therefore, we propose next a relaxed problem for (B.9).

Proposition 2. *Let the plant be as given in* (2.11)*. Let* $\Psi(\bar{Q})$ *be defined as follows*

$$\Psi(\bar{Q}) = \begin{bmatrix} z^{-1}A - I & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} z^{-1}A - I & z^{-1}AB_u \\ 0 & \epsilon_1 I \end{bmatrix} \bar{Q} \begin{bmatrix} z^{-1}A - I & 0 \\ z^{-1}C_y & \epsilon_1 I \end{bmatrix}$$

There exists an internally stabilizing controller, $K = -Q_4(I - P_{22}Q_4)^{-1}$*, if there exists a* $\bar{Q} := \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix}$ *which solves the following problem for small enough* ϵ_1 *and* ϵ_2

$$\begin{aligned} \min_{\bar{Q}} \quad & \left\| \Psi(\bar{Q}) \right\|_2^2 \\ s.t. \quad & \bar{Q} \in \mathcal{RH}_\infty. \end{aligned} \tag{B.14}$$

such that

$$\left\| \begin{bmatrix} I & 0 \end{bmatrix} - \begin{bmatrix} z^{-1}A - I & z^{-1}AB_u \end{bmatrix} \bar{Q} \right\|_2^2 + \left\| \begin{bmatrix} I \\ 0 \end{bmatrix} - \bar{Q} \begin{bmatrix} z^{-1}A - I \\ z^{-1}C_y \end{bmatrix} \right\|_2^2 < \epsilon_2. \tag{B.15}$$

Proof. Proof follows lemma 2 and corollary 14. □

Problem defined in (B.14) is an unconstrained $\mathcal{H}_2$ problem which can be solved using standard techniques to find the solution. After solving problem (B.14), one need to check if the inequality constraint given in (B.15) is satisfied. Let Q_4^* be a solution to problem (B.14) such that (B.15) is satisfied, then one can obtain the corresponding controller as $K = -Q_4^*(I - P_{22}Q_4^*)^{-1}$.

B.4 An Alternative Doubly Coprime Factorization of Controllers

In this section, we will provide an alternative doubly coprime factorization of $\bar{K}$ benefiting another stably defined plant.

Besides the stably defined congruent plant defined in (3.3a), we can obtain another stably defined congruent plant, $\hat{P}_{22}$ as in figure B.1 and its input to output maps can be given as follows

$$\hat{P}_{22} = \begin{bmatrix} z^{-1}A & z^{-1}B_u \\ z^{-1}C_yA & z^{-1}C_yB_u \end{bmatrix} \tag{B.16}$$

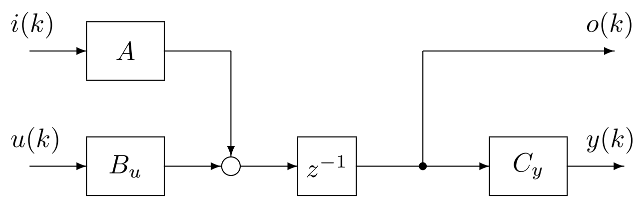

Figure B.1 Block diagram of $\hat{P}_{22}$

Then, feedback interconnection of plant P_{22} and controller K can be written as feedback interconnection of $\hat{P}_{22}$ and $\bar{K} = \textbf{blkdiag}(I_{n_x}, K)$.

Next, we show an alternative doubly coprime factorization of $\bar{K}$ benefiting definitions of $\bar{K}$ and $\hat{P}_{22}$.

Corollary 15. *Let K be a stabilizing controller of $P_{22} = ss(A, B_u, C_y, 0)$ and n_x be the order of P_{22}. Let $\bar{K}$ be defined as $\bar{K} = \begin{bmatrix} I_{n_x} & 0 \\ 0 & K \end{bmatrix}$. Define $Z = (zI - A - B_uKC_y)$ and let the set of maps M, N, $\bar{M}$ and $\bar{N}$ be defined as follows*

$$
\begin{aligned}
M = \bar{M} &= \begin{bmatrix} I + AZ^{-1} & Z^{-1}B_uK \\ KC_yZ^{-1}A & K(I - P_{22}K)^{-1} \end{bmatrix}, \\
N &= \begin{bmatrix} I + AZ^{-1} & Z^{-1}B_uK \\ C_yZ^{-1}A & (I - P_{22}K)^{-1} \end{bmatrix}, \quad \bar{N} = \begin{bmatrix} I + AZ^{-1} & Z^{-1}B_u \\ KC_yZ^{-1}A & (I - KP_{22})^{-1} \end{bmatrix}.
\end{aligned}
\tag{B.17}
$$

Then, a doubly-coprime factorization of $\bar{K}$ can be represented as $\bar{K} = MN^{-1} = \bar{N}^{-1}\bar{M}$ satisfying

$$
\Phi = \begin{bmatrix} \bar{X} & -\bar{Y} \\ -\bar{M} & \bar{N} \end{bmatrix} \begin{bmatrix} N & Y \\ M & X \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}
\tag{B.18}
$$

with stable X, Y, $\bar{X}$ and $\bar{Y}$ defined as $X = I$, $\bar{X} = I$, $Y = \bar{Y} = \begin{bmatrix} z^{-1}A & z^{-1}B_u \\ z^{-1}C_yA & z^{-1}C_yB_u \end{bmatrix}$.

Proof of corollary 15 is omitted due to its similarity to proof of lemma 7. In the next corollary, we parametrize controllers with set of maps obtain in corollary 15.

Corollary 16. *For a given plant* $P_{22} = \boldsymbol{ss}(A, B_u, C_y, 0)$*, let* $\hat{P}_{22}$ *and* M *be defined as in* (B.16) *and* (B.17)*, respectively. Then, a controller* K *of the given plant can be parametrized as*

$$K = \begin{bmatrix} 0 & I_{n_u} \end{bmatrix} M(I + \hat{P}_{22}M)^{-1} \begin{bmatrix} 0 \\ I_{n_y} \end{bmatrix}. \tag{B.19}$$

Proof. This corollary is direct result of corollary 15 and the equality $N = (I + \hat{P}_{22}M)$. □

In the next corollaries, we show that $K \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$ is network realizable controller of system $P_{22}^2 = \mathbf{ss}(A, B_u^2, C_y^2, 0) \in \mathfrak{S}(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$ wherein $A \in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_x, \mathcal{P}_x)$, $B_u^2 \in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_x, \mathcal{P}_u)$ and $C_y^2 \in S(I, \mathcal{P}_y, \mathcal{P}_x)$ are state-space matrices of P_{22}^2 and moreover, there exists a network implementable state-space realization of K over the given network.

Corollary 17. *Let* $A \in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_x, \mathcal{P}_x)$*,* $B_u^2 \in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_x, \mathcal{P}_u)$ *and* $C_y^2 \in S(I, \mathcal{P}_y, \mathcal{P}_x)$ *be state-space matrices of* P_{22}^2*, i.e.* $P_{22}^2 = \boldsymbol{ss}(A, B_u^2, C_y^2, 0) \in \mathfrak{S}(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_y, \mathcal{P}_u)$*. Let* $K \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$ *be an output feedback controller for* P_{22}^2*. Then, there exists a network implementable state space realization of* $K \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$ *over the given network.*

Proof. For a given $K \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$, M given in (B.17) belongs to set $\mathfrak{T}^s(\mathcal{G}^2, [\mathcal{P}_x; \mathcal{P}_u], [\mathcal{P}_x; \mathcal{P}_y])$. According to theorem 1, there exists a network implementable state-space realization of $M \in \mathfrak{T}^s(\mathcal{G}^2, [\mathcal{P}_x; \mathcal{P}_u], [\mathcal{P}_x; \mathcal{P}_y])$ and let $\tilde{M}$ be the network implementable state-space realization of $M \in \mathfrak{T}^s(\mathcal{G}^2, [\mathcal{P}_x; \mathcal{P}_u], [\mathcal{P}_x; \mathcal{P}_y])$. Moreover, network implementable state-space realization of $\hat{P}_{22}^2$ can be given as follows

$$\tilde{\hat{P}}_{22} = \mathbf{ss}(0, \begin{bmatrix} A & B_u \end{bmatrix}, \begin{bmatrix} I_{n_x} \\ C_y \end{bmatrix}, 0). \tag{B.20}$$

According to corollary 16, we can define the controller as in (B.19). Therefore, using network implementable state-space realizations $\tilde{M}$ and $\tilde{\hat{P}}_{22}$, we obtain the network-implementable controller, $\tilde{K}$ as

$$\tilde{K} = \begin{bmatrix} 0 & I_{n_u} \end{bmatrix} \tilde{M}(I + \tilde{\hat{P}}_{22}\tilde{M})^{-1} \begin{bmatrix} 0 \\ I_{n_y} \end{bmatrix}. \tag{B.21}$$

□

A block diagram of network implementable controller realization in (B.21) can be given as in figure B.2.

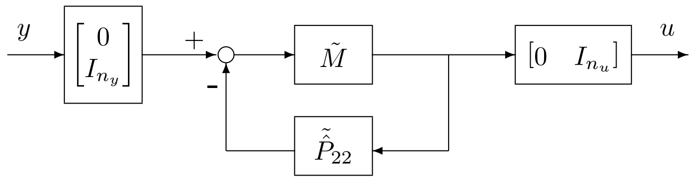

Figure B.2 A block diagram for network realization of K.

Corollary 18. *Let* $A \in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_x, \mathcal{P}_x)$*,* $B_u^2 \in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_x, \mathcal{P}_u)$ *and* $C_y^2 \in S(I, \mathcal{P}_y, \mathcal{P}_x)$ *be state-space matrices of* P_{22}*, i.e.* $P_{22}^2 = \boldsymbol{ss}(A, B_u^2, C_y^2, 0) \in \mathfrak{S}(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_y, \mathcal{P}_u)$*. Let* $K \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$ *be an output feedback controller for* P_{22}^2*. Then,* $K \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$ *is a network realizable controller of* P_{22}*.*

Proof. Since, there exists a network implementable state-space realization of controller $K \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$ of plant P_{22} according to corollary 17, it follows that K is a network realizable controller of P_{22}. □

B.5 Vectorization of Network Realizable Systems

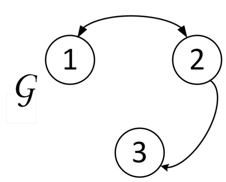

Figure B.3 A pseudo-graph of 3 node system.

For the given pseudo-graph in figure B.3, a network realizable system $Q(z) \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$ has the following structure

$$Q(z) = \begin{bmatrix} w_{11}(z) & z^{-1}w_{12}(z) & 0 \\ z^{-1}w_{21}(z) & w_{22}(z) & 0 \\ z^{-2}w_{31}(z) & z^{-1}w_{32}(z) & w_{33}(z) \end{bmatrix} \tag{B.22}$$

wherein w_{ij} for $\{i, j\} \in \{1, 2, 3\}$ are causal systems. Sparsity and delay constraints imposed on system $Q(z)$ by the set $\mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$ can be observed on (B.22). For simplicity, assume all the subsystems w_{ij} for $\{i, j\} \in \{1, 2, 3\}$ are SISO. One can give a vectorized Q as follows [37]

$$\mathbf{vec}(Q(z)) = \underbrace{\begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z^{-1}I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z^{-2}I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z^{-1}I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & z^{-1}I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}}_{S(z)} \underbrace{\begin{bmatrix} w_{11}(z) \\ w_{21}(z) \\ w_{31}(z) \\ w_{12}(z) \\ w_{22}(z) \\ w_{32}(z) \\ w_{33}(z) \end{bmatrix}}_{W(z)} = S(z)W(z) \tag{B.23}$$

Therefore, we can write $\mathbf{vec}(Q(z))$ equivalently as $S(z)W(z)$, where $S(z)$ inherits all the sparsity and delay constraints imposed by the set $\mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$.

APPENDIX C. NETWORK IMPLEMENTABLE STATE-SPACE REALIZATION OF NETWORK REALIZABLE CONTROLLERS

In this chapter we first review network implementable state-space realization technique found for stable network realizable systems, then we will show on example how to obtain network implementable state-space realization of a given network realizable controller.

C.1 Network Implementable State-Space Realization of Stable Networked Systems

In this section we review the method to find a network implementable state-space realization of given network realizable stable system as shown in [1].

In the transfer function domain a network realizable system inherits the delay and sparsity constraints of the given network. For instance the 6-node system given in figure, let P be a stable system such that $P(z) \in \mathfrak{T}^s(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$, then its transfer function inherits the following structure

$$P(z) = \begin{bmatrix} H_{11} & z^{-1}H_{12} & z^{-3}H_{13} & z^{-1}H_{14} & z^{-2}H_{15} & 0 \\ z^{-2}H_{21} & H_{22} & z^{-3}H_{23} & z^{-1}H_{24} & z^{-2}H_{25} & 0 \\ z^{-1}H_{31} & z^{-2}H_{32} & H_{33} & z^{-2}H_{34} & z^{-3}H_{35} & 0 \\ z^{-1}H_{41} & z^{-2}H_{42} & z^{-2}H_{43} & H_{44} & z^{-1}H_{45} & 0 \\ z^{-2}H_{51} & z^{-3}H_{52} & z^{-1}H_{53} & z^{-3}H_{54} & H_{55} & 0 \\ z^{-3}H_{61} & z^{-4}H_{62} & z^{-2}H_{63} & z^{-4}H_{64} & z^{-1}H_{65} & H_{66} \end{bmatrix} \quad \text{(C.1)}$$

From equation (C.1), we observe that when there is a path from node-j to i with length l (l: shortest path length), then we have $P_{ij} = z^{-l}H_{ij}$. In the case of there is no path from

i to j then we have $P_{ij} = 0$. Consider the minimal realization of P_{ij} in the following cases and define local states corresponding to a vertex as shown below.

- When $i = j$, define local state x_{ii} vertex-v_i such that

$$P_{ii}(z): \quad \begin{aligned} x_{ii}(k+1) &= A_{ii}x_{ii}(k) + B_{ii}u_i(k) \\ y_{ii}(k) &= C_{ii}x_{ii}(k) + D_{ii}u_i(k). \end{aligned} \tag{C.2}$$

- When $j \in \mathcal{N}_i^-$, define states $x_{ij}(k)$ at vertex-v_j

$$P_{ij}(z): \quad \begin{aligned} x_{ij}(k+1) &= A_{ii}x_{ij}(k) + B_{ij}u_i(k) \\ y_{ij}(k) &= C_{ii}x_{ij}(k) \end{aligned} \tag{C.3}$$

- Let l_{ij} be the shortest path from vertex v_j to v_i such that shortest path from v_j to v_i is $v_{ij}^0 \to v_{ij}^1 \to ... \to v_{ij}^{l_{ij}}$, such that $v_{ij}^0 = v_j$ and $v_{ij}^{l_{ij}} = v_i$ with intermediate vertices $v_{ij}^1, ..., v_{ij}^{l_{ij}-1}$. In this case, we define states at each vertex on the path as follows:

$$z^{-1}H_{ij}(z): \quad \begin{aligned} x_{ij}^0(k+1) &= A_{ij}x_{ij}^0(k) + B_{ij}u_j(k) \\ y_{ij}(k) &= C_{ij}x_{ij}^0(k) \end{aligned} \tag{C.4}$$

  Note that states $x_{ij}^0(k)$ are defined at vertex v_j and outputs y_{ij}^0 are passed to vertex v_{ij}^1. At vertices v_{ij}^p, $p \in \{1, ..., l-1\}$, we define states $x_{ij}^p(k)$ corresponding to unit delay systems:

$$z^{-1}: \quad \begin{aligned} x_{ij}^p(k+1) &= y_{ij}^{p-1}(k) \\ y_{ij}^p(k) &= x_{ij}^p(k) \end{aligned} \tag{C.5}$$

We denote the state vector corresponding to each vertex v_i to be $\tilde{x}_i(k)$, which is formed by appending the states $x_{ii}(k)$, $x_{ri}(k)$ $\forall r \in \mathcal{N}_i^+$ and $x_{mn}^p(k)$ whenever $v_{mn}^p = v_i$ (for $p \in \{0, ..., l_{mn} - 1\}$), i.e. when vertex v_i is a vertex on the shortest path from some vertex v_n to some other vertex v_a.

A network output vector η_{ri}, for all $r \in \mathcal{N}_i^+$, is formed by appending $y_{ri}(k)$ and $y_{ab}^p(k)$ whenever $v_{mn}^p = v_i$ and $v_{mn}^{p+1} = v_r$ (for $p \in \{0, 1, ..., l_{mn} - 1\}$). Similarly, a network input vector $\tilde{\zeta}(k)$, for all $j \in \mathcal{N}_i^-$, is formed by appending $y_{ij}(k)$ and $y_{mn}^p(k)$ whenever $v_{mn}^p = v_j$

and $v_{mn}^{p+1} = v_i$ (for $p \in \{0, ..., l_{mn} - 1\}$). Note that the network inputs defined at vertex v_i do not affect the outputs at the same vertex v_i for any time instant k.

At vertex v_i, the output $y_i(k)$ is given by

$$y_i(k) = y_{ii} + \sum_{j \in \mathcal{N}_i^-} y_{ij}(k) + \sum_{j: l_{ij} \geq 2} y_{ij}^{l_{ij}-1}(k). \tag{C.6}$$

Thus, we can define n sub-systems, $\{\tilde{P}_i\}_i$, each with local states $\tilde{x}_i(k)$, local inputs $u_i(k)$, local outputs $y_i(k)$, network inputs $\tilde{\zeta}_{ij}(k)$ (for all $j \in \mathcal{N}_i^-$) and network outputs $\tilde{\eta}_{ir}(k)$ (for all $r \in \mathcal{N}_i^+$).

C.2 A Network Implementable State-Space Realization of Network Realizable Controller

In this section we will demonstrate how to obtain network implementable state-space realization of a network realizable controller given for 6-node system given in figure 2.1 by benefiting the network implementable state-space realization method reviewed in section C.1.

Let V be defined as in (4.2) and $\bar{P}_{22}$ be defined as in (3.3). We will obtain network implementable state-space realization of network realizable controlller, $K \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$, by obtaining network implementable state space realization of $V(I + \bar{P}_{22}\bar{V})^{-1}$ as shown in chapter 4. As it can be seen in figure 4.1, we can do realization of $V(I + \bar{P}_{22}\bar{V})^{-1}$ by two blocks $\tilde{V}$ and $\tilde{\bar{P}}_{22}$ which are network implementable state-space realizations of V and $\bar{P}_{22}$. A network implementable state-space realization of $\bar{P}_{22}$ can be found in (4.6). Next, we will demonstrate how to obtain network implementable state-space realization of $V \in \mathfrak{T}^s(\mathcal{G}^2, [\mathcal{P}_x; \mathcal{P}_u], [\mathcal{P}_x; \mathcal{P}_y])$.

Using definition of V, let us define its sub-blocks:

$$V := \begin{bmatrix} V_1 & V_2 \\ V_3 & V_4 \end{bmatrix} = \begin{bmatrix} I + AZ^{-1} & AZ^{-1}B_uK \\ KC_yZ^{-1} & K(I - P_{22}K)^{-1} \end{bmatrix} \tag{C.7}$$

For the six-node system given in figure 2.1, in z-domain each V_i for $i = \{1, ..., 4\}$ has the following form:

$$V_i = \begin{bmatrix} V_i^{11} & z^{-1}V_i^{12} & z^{-3}V_i^{13} & z^{-1}V_i^{14} & z^{-2}V_i^{15} & 0 \\ z^{-2}V_i^{21} & V_i^{22} & z^{-3}V_i^{23} & z^{-1}V_i^{24} & z^{-2}V_i^{25} & 0 \\ z^{-1}V_i^{31} & z^{-2}V_i^{32} & V_i^{33} & z^{-2}V_i^{34} & z^{-3}V_i^{35} & 0 \\ z^{-1}V_i^{41} & z^{-2}V_i^{42} & z^{-2}V_i^{43} & V_i^{44} & z^{-1}V_i^{45} & 0 \\ z^{-2}V_i^{51} & z^{-3}V_i^{52} & z^{-1}V_i^{53} & z^{-3}V_i^{54} & V_i^{55} & 0 \\ z^{-3}V_i^{61} & z^{-4}V_i^{62} & z^{-2}V_i^{63} & z^{-4}V_i^{64} & z^{-1}V_i^{65} & V_i^{66} \end{bmatrix} \tag{C.8}$$

where V_i^{kj} are casual stable systems.

Let $\hat{V}$ be a matrix such that its input and outputs are regrouped form of V as follows:

$$\hat{V} := \begin{bmatrix} \hat{V}_{11} & \hat{V}_{12} & \hat{V}_{13} & \hat{V}_{14} & \hat{V}_{15} & \hat{V}_{16} \\ \hat{V}_{21} & \hat{V}_{22} & \hat{V}_{23} & \hat{V}_{24} & \hat{V}_{25} & \hat{V}_{26} \\ \hat{V}_{31} & \hat{V}_{32} & \hat{V}_{33} & \hat{V}_{34} & \hat{V}_{35} & \hat{V}_{36} \\ \hat{V}_{41} & \hat{V}_{42} & \hat{V}_{43} & \hat{V}_{44} & \hat{V}_{45} & \hat{V}_{46} \\ \hat{V}_{51} & \hat{V}_{52} & \hat{V}_{53} & \hat{V}_{54} & \hat{V}_{55} & \hat{V}_{56} \\ \hat{V}_{11} & \hat{V}_{12} & \hat{V}_{13} & \hat{V}_{14} & \hat{V}_{15} & \hat{V}_{16} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} V_1^{11} & V_2^{11} \\ V_3^{11} & V_4^{11} \end{bmatrix} & \cdots & z^{-2}\begin{bmatrix} V_1^{15} & V_2^{15} \\ V_3^{15} & V_4^{15} \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \vdots & \ddots & \vdots & \vdots \\ z^{-2}\begin{bmatrix} V_1^{51} & V_2^{51} \\ V_3^{51} & V_4^{51} \end{bmatrix} & \cdots & \begin{bmatrix} V_1^{55} & V_2^{55} \\ V_3^{55} & V_4^{55} \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ z^{-3}\begin{bmatrix} V_1^{61} & V_2^{61} \\ V_3^{61} & V_4^{61} \end{bmatrix} & \cdots & z^{-1}\begin{bmatrix} V_1^{65} & V_2^{65} \\ V_3^{65} & V_4^{65} \end{bmatrix} & \begin{bmatrix} V_1^{66} & V_2^{66} \\ V_3^{66} & V_4^{66} \end{bmatrix} \end{bmatrix} \tag{C.9}$$

Let $\hat{V}_{ij}(z)$ for $\{i, j\} \in \{1, 2, ..., 6\}$ be the entries of $\hat{V}(z)$ as shown in (C.9). As shown in equation (C.9), columns of $\hat{V}(z)$ includes terms, $\hat{V}_{ii}(z)$ and $z^{-k}\hat{V}_{ij}(z)$ for some $k \in \{1, 2, 3, 4\}$. Note that $z^{-k}\hat{V}_{ij}$ is essentially the transfer function matrix mapping $u_j(k)$ to

$y_i(k)$. System $\hat{V}_{ij}(z)$ with z^{-1}, i.e. $z^{-1}\hat{V}_{ij}(z)$, shows that messages of system $\hat{V}_{ij}(z)$ goes directly from node i to node j. On the other side, systems like $z^{-k}\hat{V}_{ij}(z)$ for $k > 1$ shows that messages of system $\hat{V}_{ij}$ does not directly go from node i to node j. For instance, we have $z^{-2}\hat{V}_{34}(z)$, in this case message from node 4 to node 3 goes through node 1 (see figure 2.1).

Let $G_j(z)$ for $\{1, 2..., 6\}$ be the column matrix obtained using jth column entries of $\hat{V}(z)$

$$G_1(z) = \begin{bmatrix} \hat{V}_{11}(z) \\ z^{-1}\hat{V}_{21}(z) \\ z^{-1}\hat{V}_{31}(z) \\ z^{-1}\hat{V}_{41}(z) \\ z^{-1}\hat{V}_{51}(z) \\ z^{-1}\hat{V}_{61}(z) \end{bmatrix}, G_2(z) = \begin{bmatrix} z^{-1}\hat{V}_{12}(z) \\ \hat{V}_{22}(z) \\ z^{-1}\hat{V}_{32}(z) \\ z^{-1}\hat{V}_{42}(z) \\ z^{-1}\hat{V}_{52}(z) \\ z^{-1}\hat{V}_{62}(z) \end{bmatrix}, G_3 = \begin{bmatrix} z^{-1}\hat{V}_{13}(z) \\ z^{-1}\hat{V}_{23}(z) \\ \hat{V}_{33}(z) \\ z^{-1}\hat{V}_{43}(z) \\ z^{-1}\hat{V}_{53}(z) \\ z^{-1}\hat{V}_{63}(z) \end{bmatrix}$$

$$G_4(z) = \begin{bmatrix} z^{-1}\hat{V}_{14}(z) \\ z^{-1}\hat{V}_{24}(z) \\ z^{-1}\hat{V}_{34}(z) \\ \hat{V}_{44}(z) \\ z^{-1}\hat{V}_{54}(z) \\ z^{-1}\hat{V}_{64}(z) \end{bmatrix}, G_5(z) = \begin{bmatrix} z^{-1}\hat{V}_{15}(z) \\ z^{-1}\hat{V}_{25}(z) \\ z^{-1}\hat{V}_{35}(z) \\ z^{-1}\hat{V}_{45}(z) \\ \hat{V}_{55}(z) \\ z^{-1}\hat{V}_{65}(z) \end{bmatrix}, G_6 = \begin{bmatrix} \hat{V}_{66}(z) \end{bmatrix}.$$

Minimal state space realizations of G_j for $j \in \{1, 2, ..., 6\}$ can be shown as

$$G_j(z) \rightarrow \begin{bmatrix} x_{G_j}(k+1) \\ \hline y_{G_j}(k) \end{bmatrix} = \left[\begin{array}{c|c} A_{G_j} & B_{G_j} \\ \hline C_{G_j} & D_{G_j} \end{array}\right] \begin{bmatrix} x_{G_j}(k) \\ \hline \tilde{u}_j(k) \end{bmatrix} \tag{C.10}$$

and corresponding output vectors are

$$y_{G_1} = \begin{bmatrix} y_{11}^T & (y_{21}^0)^T & y_{31}^T & y_{41}^T & (y_{51}^0)^T & (y_{61}^0)^T \end{bmatrix}^T,$$

$$y_{G_2} = \begin{bmatrix} y_{12}^T & y_{22}^T & (y_{32}^0)^T & (y_{42}^0)^T & (y_{52}^0)^T & (y_{62}^0)^T \end{bmatrix}^T,$$

$$y_{G_3} = \begin{bmatrix} (y_{13}^0)^T & (y_{23}^0)^T & y_{33}^T & (y_{43}^0)^T & y_{53}^T & (y_{63}^0)^T \end{bmatrix}^T$$

,

$$y_{G_4} = \begin{bmatrix} y_{14}^T & y_{24}^T & (y_{34}^0)^T & y_{44}^T & (y_{54}^0)^T & (y_{64}^0)^T \end{bmatrix}^T,$$

$$y_{G_5} = \begin{bmatrix} (y_{15}^0)^T & (y_{25}^0)^T & (y_{35}^0)^T & y_{45}^T & y_{55}^T & y_{65}^T \end{bmatrix}^T,$$

$$y_{G_6} = \begin{bmatrix} (y_{66})^T \end{bmatrix}^T.$$

Moreover, for nodes $1-5$, we can write the equality y_{G_j} explicitly as

$$y_{G_j} = \begin{bmatrix} C_{G_j}^1 \\ C_{G_j}^2 \\ C_{G_j}^3 \\ C_{G_j}^4 \\ C_{G_j}^5 \\ C_{G_j}^6 \end{bmatrix} x_{G_j} + \begin{bmatrix} D_{G_j}^1 \\ D_{G_j}^2 \\ D_{G_j}^3 \\ D_{G_j}^4 \\ D_{G_j}^5 \\ D_{G_j}^6 \end{bmatrix} u_{G_j} \tag{C.11}$$

We define the $y_{34}^0(k)$ at node 4, and its output passes to node 1 as input of unit delay system. Therefore, at node 1, we define the states $x_{34}^1(k)$ corresponding to unit delay systems. Similarly, other delayed systems can be defined as in the following where $(v_1, v_2) \in \{(3,2),(5,2),(6,2),(3,4),(5,4),(6,4),(4,2),(5,1),(6,1),(1,5),(3,5),(2,1),(2,5),(1,3),$ $(2,3),(4,3),(6,3)\}$, $(v_3, v_4) \in \{(3,5),(5,2),(6,2),(5,4),(6,4),(1,3),(2,3),(6,1),(6,3)\}$, and $(v_5, v_6) \in \{(6,2),(6,4)\}$.

$$\begin{aligned} z^{-1} \rightarrow \left[\begin{array}{c} x_{v_1,v_2}^1(k+1) \\ \hline y_{v_1,v_2}^1(k) \end{array}\right] &= \left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right] \left[\begin{array}{c} x_{v_1,v_2}^1(k) \\ \hline y_{v_1,v_2}^0(k) \end{array}\right] \\ z^{-1} \rightarrow \left[\begin{array}{c} x_{v_3,v_4}^2(k+1) \\ \hline y_{v_3,v_4}^2(k) \end{array}\right] &= \left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right] \left[\begin{array}{c} x_{v_3,v_4}^2(k) \\ \hline y_{v_3,v_4}^1(k) \end{array}\right] \\ z^{-1} \rightarrow \left[\begin{array}{c} x_{v_5,v_6}^3(k+1) \\ \hline y_{v_5,v_6}^3(k) \end{array}\right] &= \left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right] \left[\begin{array}{c} x_{v_5,v_6}^3(k) \\ \hline y_{v_6,v_6}^2(k) \end{array}\right] \end{aligned} \tag{C.12}$$

By regarding the information flow in the network, state vectors corresponding to each node can be given as in the followings

$$\tilde{x}_1(k) = \begin{bmatrix} x_{G_1}(k) \\ x^1_{32}(k) \\ x^1_{52}(k) \\ x^1_{62}(k) \\ x^1_{34}(k) \\ x^1_{54}(k) \\ x^1_{64}(k) \\ x^2_{35}(k) \\ x^1_{42}(k) \end{bmatrix}, \tilde{x}_3(k) = \begin{bmatrix} x_{G_3}(k) \\ x^1_{51}(k) \\ x^1_{61}(k) \\ x^2_{52}(k) \\ x^2_{62}(k) \\ x^2_{54}(k) \\ x^2_{64}(k) \end{bmatrix}, \tilde{x}_4(k) = \begin{bmatrix} x_{G_4}(k) \\ x^2_{13}(k) \\ x^1_{15}(k) \\ x^1_{35}(k) \\ x^1_{21}(k) \\ x^2_{23}(k) \\ x^1_{25}(k) \end{bmatrix}, \tilde{x}_5(k) = \begin{bmatrix} x_{G_5}(k) \\ x^1_{13}(k) \\ x^1_{23}(k) \\ x^1_{43}(k) \\ x^2_{61}(k) \\ x^3_{62}(k) \\ x^1_{63}(k) \\ x^3_{64}(k) \end{bmatrix}$$

$\tilde{x}_2(k) = x_{G_2}(k)$ and $\tilde{x}_6(k) = x_{G_6}(k)$.

We denote outgoing messages from node-i to node-j as $\tilde{\eta}_{ji}$. Outgoing messages can be given as

$$\tilde{\eta}_{12}(k) = \begin{bmatrix} y^0_{32}(k) \\ y^0_{52}(k) \\ y^0_{62}(k) \\ y^0_{42}(k) \\ y_{12}(k) \end{bmatrix}, \tilde{\eta}_{31}(k) = \begin{bmatrix} y^1_{32}(k) \\ y^1_{52}(k) \\ y^1_{62}(k) \\ y^1_{34}(k) \\ y^1_{54}(k) \\ y^1_{64}(k) \\ y^2_{35}(k) \\ y_{31}(k) \\ y^0_{51}(k) \\ y^0_{61}(k) \end{bmatrix}, \tilde{\eta}_{53}(k) = \begin{bmatrix} y^1_{51}(k) \\ y^1_{61}(k) \\ y^2_{52}(k) \\ y^2_{62}(k) \\ y^2_{54}(k) \\ y^2_{64}(k) \\ y^0_{13}(k) \\ y^0_{23}(k) \\ y^0_{43}(k) \\ y^0_{63}(k) \\ y_{53}(k) \end{bmatrix}, \tilde{\eta}_{14}(k) = \begin{bmatrix} y^2_{13}(k) \\ y^1_{15}(k) \\ y^1_{35}(k) \\ y^0_{34}(k) \\ y^0_{54}(k) \\ y^0_{64}(k) \\ y_{14}(k) \end{bmatrix},$$

$$\tilde{\eta}_{24}(k)=\begin{bmatrix}y_{21}^{1}(k)\\ y_{23}^{2}(k)\\ y_{25}^{1}(k)\\ y_{24}(k)\end{bmatrix},\tilde{\eta}_{41}(k)=\begin{bmatrix}y_{42}^{1}(k)\\ y_{41}(k)\\ y_{21}^{0}(k)\end{bmatrix},\tilde{\eta}_{45}(k)=\begin{bmatrix}y_{13}^{1}(k)\\ y_{23}^{1}(k)\\ y_{43}^{1}(k)\\ y_{15}^{0}(k)\\ y_{35}^{0}(k)\\ y_{25}^{0}(k)\\ y_{45}(k)\end{bmatrix},\tilde{\eta}_{65}(k)=\begin{bmatrix}y_{61}^{2}(k)\\ y_{62}^{3}(k)\\ y_{63}^{1}(k)\\ y_{64}^{3}(k)\\ y_{65}(k)\end{bmatrix}.$$

Furthermore, outputs at each node can be given as

$$\begin{aligned}
\tilde{y}_1(k) &= y_{11}(k)+y_{12}(k)+y_{13}^{2}(k)+y_{14}(k)+y_{15}^{1}(k),\\
\tilde{y}_2(k) &= y_{21}^{1}(k)+y_{22}(k)+y_{23}^{2}(k)+y_{24}(k)+y_{25}^{1}(k),\\
\tilde{y}_3(k) &= y_{31}(k)+y_{32}^{1}(k)+y_{33}(k)+y_{34}^{1}(k)+y_{35}^{2}(k),\\
\tilde{y}_4(k) &= y_{41}(k)+y_{42}^{1}(k)+y_{43}^{1}(k)+y_{44}(k)+y_{45}(k),\\
\tilde{y}_5(k) &= y_{51}^{1}(k)+y_{52}^{2}(k)+y_{53}(k)+y_{54}^{2}(k)+y_{55}(k),\\
\tilde{y}_6(k) &= y_{61}^{2}(k)+y_{62}^{3}(k)+y_{63}^{1}(k)+y_{64}^{3}(k)+y_{65}(k)+y_{66}(k).
\end{aligned} \tag{C.13}$$

Since the network $\mathcal{G}$ is noiseless and has zero delay, the incoming message vectors at each vertex are given by

$$\begin{aligned}
&\tilde{\zeta}_{31}(k)=\tilde{\eta}_{31}(k),\quad \tilde{\zeta}_{41}(k)=\tilde{\eta}_{41}(k),\quad \tilde{\zeta}_{53}(k)=\tilde{\eta}_{53}(k),\quad \tilde{\zeta}_{14}(k)=\tilde{\eta}_{14}(k),\\
&\tilde{\zeta}_{24}(k)=\tilde{\eta}_{24}(k),\quad \tilde{\zeta}_{45}(k)=\tilde{\eta}_{45}(k),\quad \tilde{\zeta}_{65}(k)=\tilde{\eta}_{65}(k).
\end{aligned} \tag{C.14}$$

Using equations (C.10), (C.11), (C.12), (C.13) and (C.14) the dynamics at each vertex can be defined as a sub-system $\tilde{V}_i$ for $i=\{1,...,6\}$. Subsystems $\tilde{V}_1$, $\tilde{V}_2$, $\tilde{V}_3$, $\tilde{V}_4$ and $\tilde{V}_5$ can be given as

$$
\left[\begin{array}{c}\tilde{x}_1(k+1)\\ \hline \tilde{y}_1(k)\\ \hline \tilde{\eta}_{31}(k)\\ \tilde{\eta}_{41}(k)\end{array}\right]
=
\left[\begin{array}{ccccccccc|c|cccccccccccc}
A_{G_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{G_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\hline
C^1_{G_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D^1_{G_1} & 0 & 0 & 0 & 0 & I & I & I & 0 & 0 & 0 & 0 & I\\
\hline
0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
C^3_{G_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
C^5_{G_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
C^6_{G_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
C^4_{G_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
C^2_{G_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\left[\begin{array}{c}\tilde{x}_1(k)\\ \hline \tilde{u}_1(k)\\ \hline \tilde{\zeta}_{12}(k)\\ \tilde{\zeta}_{14}(k)\end{array}\right]
\tag{C.15}
$$

$$
\left[\begin{array}{c}\tilde{x}_2(k+1)\\ \hline \tilde{y}_2(k)\\ \hline \tilde{\eta}_{12}(k)\end{array}\right]
=
\left[\begin{array}{c|c|cccc}
A_{G_2} & B_{G_2} & 0 & 0 & 0 & 0\\
\hline
C^2_{G_2} & D^2_{G_2} & I & I & I & I\\
\hline
C^3_{G_2} & 0 & 0 & 0 & 0 & 0\\
C^5_{G_2} & 0 & 0 & 0 & 0 & 0\\
C^6_{G_2} & 0 & 0 & 0 & 0 & 0\\
C^4_{G_2} & 0 & 0 & 0 & 0 & 0\\
C^1_{G_2} & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\left[\begin{array}{c}\tilde{x}_2(k)\\ \hline \tilde{u}_2(k)\\ \hline \tilde{\zeta}_{24}(k)\end{array}\right]
\tag{C.16}
$$

$$
\left[\begin{array}{c}
\tilde{x}_3(k+1) \\ \hline
\tilde{y}_1(k) \\ \hline
\tilde{\eta}_{53}(k)
\end{array}\right]
=
\left[\begin{array}{ccccccc|c|cccccccccc}
A_{G_3} & 0 & 0 & 0 & 0 & 0 & 0 & B_{G_3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ \hline
C^3_{G_3} & 0 & 0 & 0 & 0 & 0 & 0 & D^3_{G_3} & I & 0 & 0 & I & 0 & 0 & I & I & 0 & 0 \\ \hline
0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C^1_{G_3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C^2_{G_3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C^4_{G_3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C^6_{G_3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C^5_{G_3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\left[\begin{array}{c}
\tilde{x}_3(k) \\ \hline
\tilde{u}_3(k) \\ \hline
\tilde{\zeta}_{31}(k)
\end{array}\right]
\tag{C.17}
$$

$$
\left[\begin{array}{c}
\tilde{x}_4(k+1) \\ \hline
\tilde{y}_4(k) \\ \hline
\tilde{\eta}_{14}(k) \\
\tilde{\eta}_{24}(k)
\end{array}\right]
=
\left[\begin{array}{ccccccc|c|cccccccccc}
A_{G_4} & 0 & 0 & 0 & 0 & 0 & 0 & B_{G_4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ \hline
C_{G_4}^4 & 0 & 0 & 0 & 0 & 0 & 0 & D_{G_4}^4 & I & I & 0 & 0 & 0 & 0 & 0 & 0 & I & I \\ \hline
0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_{G_4}^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_{G_4}^5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_{G_4}^6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_{G_4}^1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_{G_3}^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\left[\begin{array}{c}
\tilde{x}_4(k) \\ \hline
\tilde{u}_4(k) \\ \hline
\tilde{\zeta}_{41}(k) \\
\tilde{\zeta}_{45}(k)
\end{array}\right]
\tag{C.18}
$$

$$
\left[\begin{array}{c}
\tilde{x}_5(k+1) \\ \hline
\tilde{y}_5(k) \\ \hline
\tilde{\eta}_{45}(k) \\
\tilde{\eta}_{65}(k)
\end{array}\right]
=
\left[\begin{array}{cccccccc|c|ccccccccccc}
A_{G_5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{G_5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ \hline
C_{G_5}^5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_{G_5}^5 & I & 0 & I & 0 & I & 0 & 0 & 0 & 0 & 0 & I \\ \hline
0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_{G_5}^1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_{G_5}^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_{G_5}^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_{G_5}^4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_{G_5}^6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\left[\begin{array}{c}
\tilde{x}_5(k) \\ \hline
\tilde{u}_5(k) \\ \hline
\tilde{\zeta}_{53}(k)
\end{array}\right]
\tag{C.19}
$$

Finally, sub-system $\tilde{V}_6$ can be given as

$$
\left[\begin{array}{c}
\tilde{x}_6(k+1) \\ \hline
\tilde{y}_6(k)
\end{array}\right]
=
\left[\begin{array}{c|c|ccccc}
A_{G_6} & B_{G_6} & 0 & 0 & 0 & 0 & 0 \\ \hline
C_{G_6} & D_{G_6} & I & I & I & I & I
\end{array}\right]
\left[\begin{array}{c}
\tilde{x}_6(k) \\ \hline
\tilde{u}_6(k) \\ \hline
\tilde{\zeta}_{65}(k)
\end{array}\right]
\tag{C.20}
$$

The subsystems $\tilde{V}_i$ $i \in \{1, ..., 6\}$ given by (C.15), (C.16), (C.17), (C.18), (C.19) and (C.20) interacting over the interconnection (C.14) describes the networked system $\tilde{\hat{V}}$ corresponding to $\hat{V}(z)$. Combining equations in (C.14), (C.15), (C.16), (C.17), (C.18), (C.19) and (C.20), one can obtain state space representation of $\hat{V}$ as $\tilde{\hat{V}} = \mathbf{ss}(A_{\hat{V}}, B_{\hat{V}}, C_{\hat{V}}, D_{\hat{V}}) \in$

$\mathfrak{S}(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$ such that $A_{\hat{V}} \in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_{x_{\bar{V}}}, \mathcal{P}_{x_{\bar{V}}})$, $B_{\hat{V}} \in S(I, \mathcal{P}_{x_{\bar{V}}}, \mathcal{P}_{u_{\bar{V}}})$, $C_{\hat{V}} \in S(\mathcal{A}(\mathcal{G}), \mathcal{P}_{y_{\bar{V}}}, \mathcal{P}_{x_{\bar{V}}})$ and $D_{\hat{V}} \in S(I, \mathcal{P}_{y_{\bar{V}}}, \mathcal{P}_{u_{\bar{V}}})$ where $\mathcal{P}_{y_{\bar{V}}} = (3,3,3,3,3,3)$, $\mathcal{P}_{u_{\bar{V}}} = (3,3,3,3,3,3)$ and $\mathcal{P}_{x_{\bar{V}}} = (n_1, n_2, n_3, n_4, n_5, n_6)$ wherein n_i is length of $\tilde{x}_i$ for $i \in \{1, ..., 6\}$.

Since $\hat{V}$ is obtained from V by regrouping the inputs and outputs, one can obtain back network implementable state-space realization of $\tilde{V}$ from $\tilde{\bar{V}}$ by re-grouping its inputs and outputs channels. So, we can obtain network implementable state-space realization of V as $\bar{V} = \mathbf{ss}(A_V, B_V, C_V, D_V) \in \mathfrak{S}(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$ such that $A_V = A_{\hat{V}}$, B_V is obtained from $B_{\hat{V}}$ by a proper row exchanges, C_V is obtained from $C_{\hat{V}}$ by a proper column exchanges, and D_V is obtained from $D_{\hat{V}}$ by a proper row and column exchanges (proper row and column exchanges are referred to row and column exchanges transforms $\hat{V}$ back to V), so we have $\mathbf{tf}(\tilde{V}) = V(z)$.

Using network implementable state space realizations $\tilde{V} = \mathbf{ss}(A_V, B_V, C_V, D_V)$ and $\tilde{\bar{P}}_{22} = \mathbf{ss}(A_{\bar{P}_{22}}, B_{\bar{P}_{22}}, C_{\bar{P}_{22}}, D_{\bar{P}_{22}})$ ($\tilde{\bar{P}}_{22}$ can be find as in (4.6)) we obtain network implementable state space realization of $\bar{K} = \mathbf{blkdiag}(I, K)$ using block diagram 4.1 and its state-space matrices can be given as

$$
\begin{aligned}
\tilde{\bar{K}} &= \left[\begin{array}{c|c} A_{\bar{K}} & B_{\bar{K}} \\ \hline C_{\bar{K}} & D_{\bar{K}} \end{array}\right] \\
A_{\bar{K}} &:= \begin{bmatrix} A_{\bar{P}_{22}} - B_{\bar{P}_{22}} D_V C_{\bar{P}_{22}} & B_{\bar{P}_{22}} C_V \\ -B_V C_{\bar{P}_{22}} & A_V \end{bmatrix}, \\
B_{\bar{K}} &:= \begin{bmatrix} B_{\bar{P}_{22}} D_V \\ B_V \end{bmatrix}, \\
C_{\bar{K}} &:= \begin{bmatrix} -D_V C_{\bar{P}_{22}} & C_V \end{bmatrix}, \\
D_{\bar{K}} &:= D_V.
\end{aligned}
\tag{C.21}
$$

Using $\bar{K}$, we can obtain K as $K = \begin{bmatrix} 0 & I_{n_u} \end{bmatrix} \bar{K} \begin{bmatrix} 0 \\ I_{n_y} \end{bmatrix}$. Therefore, using (C.21), network implementable state-space realization of $\tilde{\bar{K}}$ can be given as

$$
\begin{aligned}
\tilde{K} &= \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right] \\
A_K &:= \begin{bmatrix} A_{\bar{P}_{22}} - B_{\bar{P}_{22}} D_V C_{\bar{P}_{22}} & B_{\bar{P}_{22}} C_V \\ -B_V C_{\bar{P}_{22}} & A_V \end{bmatrix}, \\
B_K &:= \begin{bmatrix} B_{\bar{P}_{22}} D_V \\ B_V \end{bmatrix} \begin{bmatrix} 0 \\ I_{n_y} \end{bmatrix}, \\
C_K &:= \begin{bmatrix} 0 & I_{n_u} \end{bmatrix} \begin{bmatrix} -D_V C_{\bar{P}_{22}} & C_V \end{bmatrix}, \\
D_K &:= \begin{bmatrix} 0 & I_{n_u} \end{bmatrix} D_V \begin{bmatrix} 0 \\ I_{n_y} \end{bmatrix}.
\end{aligned}
\tag{C.22}
$$